Example 13

Isao Sasano

Example The Fourier series $f(x) = x^2$ on the range $[-\pi, \pi]$ is

$$\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \cos kx. \tag{1}$$

Rewrite this series in the form of a linear combination of complex exponential functions $\{e^{ikx} | k \in \mathbb{Z}\}$.

Solution By the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2}$$

the following equation holds.

$$\begin{aligned} e^{-i\theta} &= e^{i(-\theta)} \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned} \tag{3}$$

By adding the equations (2) and (3) we obtain the following equation.

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

By subtracting the equation (3) from (2) we obtain the following equation.

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

So we obtain the following equations.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{4}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

By substituting kx for θ in the equation (4) we obtain the following equation.

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}$$

By setting $\cos kx = \dfrac{e^{ikx} + e^{-ikx}}{2}$ in (1) we obtain the following series.

$$\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \frac{e^{ikx} + e^{-ikx}}{2}$$

We rewrite this series as follows.

$$\begin{aligned}
&\frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \frac{e^{ikx} + e^{-ikx}}{2} \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \frac{2}{k^2}(-1)^k (e^{ikx} + e^{-ikx}) \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \left\{ \frac{2}{k^2}(-1)^k e^{ikx} + \frac{2}{k^2}(-1)^k e^{-ikx} \right\} \\
&= \frac{\pi^2}{3} + \sum_{k=1}^{\infty} \left\{ \frac{2}{k^2}(-1)^k e^{ikx} + \frac{2}{(-k)^2}(-1)^{(-k)} e^{i(-k)x} \right\} \\
&= \frac{\pi^2}{3} + \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ \frac{2}{k^2}(-1)^k e^{ikx} + \frac{2}{(-k)^2}(-1)^{(-k)} e^{i(-k)x} \right\} \\
&= \frac{\pi^2}{3} + \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{2}{k^2}(-1)^k e^{ikx} + \sum_{k=-n}^{-1} \frac{2}{k^2}(-1)^k e^{ikx} \right\} \\
&= \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx}
\end{aligned}$$

Here c_k is defined as follows.

$$c_k = \begin{cases} \dfrac{2}{k^2}(-1)^k & k > 0 \\ \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2}{k^2}(-1)^k & k < 0 \end{cases}$$

Note that $\lim_{n\to\infty} \sum_{k=-n}^{n} a_k$ is usually written as $\sum_{k=-\infty}^{\infty} a_k$. So the above expression $\lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx}$ can be written as follows.

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$