Exercise 9-1

Isao Sasano

Exercise Consider the set of real-valued continuous functions on the interval $[0,1]$. Then the following two operations, the addition and scalar multiplication, satisfy the axioms of linear space (although I have not fully explained the axioms for linear space in this class).

$$\begin{aligned}(\boldsymbol{f}+\boldsymbol{g})(x) &= f(x)+g(x)\\ (c\boldsymbol{f})(x) &= c(f(x))\end{aligned}$$

On this linear space we consider the following operation.

$$(\boldsymbol{f},\boldsymbol{g}) = \int_0^1 f(x)g(x)\mathrm{d}x$$

We restrict the functions in the space to be the ones so that for any two functions in the space they have a value of the operation[1]. Then this operation satisfies the axioms for inner product. By introducing the above operation as the inner product into this linear space, we obtain an inner product space.

On this inner product space, calculate the inner product of $f_1(x) = x$ and $f_2(x) = x^2$ and the product of those norm and compare them.

Solution

$$\begin{aligned}(\boldsymbol{f}_1,\boldsymbol{f}_2) &= \int_0^1 f_1(x)f_2(x)\mathrm{d}x\\ &= \int_0^1 x^3\mathrm{d}x\end{aligned}$$

[1]It also depends on the definition of the integral whether or not the integral has a value. The definitions of integral include the Riemann integral, which we usually learn in the analysis, and the Lebesgue integral, which we learn in the department of mathematics, and so on. Actually, in the Riemann integral and in the Lebesgue integral, the integral of any continuous function on any closed bounded interval has a value, so under these definitions of integral we do not need to restrict functions.

$$
\begin{aligned}
&= \left[\frac{x^4}{4}\right]_0^1 \\
&= \frac{1}{4} \\
\|\boldsymbol{f}_1\| &= \sqrt{(\boldsymbol{f}_1, \boldsymbol{f}_1)} \\
&= \sqrt{\int_0^1 x^2 \mathrm{d}x} \\
&= \sqrt{\left[\frac{x^3}{3}\right]_0^1} \\
&= \sqrt{\frac{1}{3}} \\
\|\boldsymbol{f}_2\| &= \sqrt{(\boldsymbol{f}_2, \boldsymbol{f}_2)} \\
&= \sqrt{\int_0^1 x^4 \mathrm{d}x} \\
&= \sqrt{\left[\frac{x^5}{5}\right]_0^1} \\
&= \sqrt{\frac{1}{5}}
\end{aligned}
$$

We compare the inner product $(\boldsymbol{f}_1, \boldsymbol{f}_2)$ and the product of those norm $\|\boldsymbol{f}_1\| \cdot \|\boldsymbol{f}_2\|$.

$$
\begin{aligned}
\|\boldsymbol{f}_1\| \cdot \|\boldsymbol{f}_2\| &= \sqrt{\frac{1}{3}} \cdot \sqrt{\frac{1}{5}} \\
&= \sqrt{\frac{1}{15}} \\
&\geq \sqrt{\frac{1}{16}} \\
&= \frac{1}{4} \\
&= (\boldsymbol{f}_1, \boldsymbol{f}_2)
\end{aligned}
$$

So we have shown that the Cauchy-Schwarz inequality holds for the above example.