Solutions for Exercise 5-2

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Exercise Calculate $P_4(x)$ from the general form of Legendre polynomials.

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2 - 1)^n$$

Solution

(Solution 1)

$$\begin{aligned}
P_4(x) &= \frac{1}{2^4 \cdot 4!} \frac{\mathrm{d}^4}{\mathrm{d}x^4} (x^2 - 1)^4 \\
&= \frac{1}{2^4 \cdot 4!} \frac{\mathrm{d}^4}{\mathrm{d}x^4} (x^4 - 2x^2 + 1)^2 \\
&= \frac{1}{2^4 \cdot 4!} \frac{\mathrm{d}^4}{\mathrm{d}x^4} (x^8 - 4x^6 + 6x^4 - 4x^2 + 1) \\
&= \frac{1}{2^4 \cdot 4!} \frac{\mathrm{d}^3}{\mathrm{d}x^3} (8x^7 - 4 \cdot 6x^5 + 6 \cdot 4x^3 - 4 \cdot 2x) \\
&= \frac{1}{2^4 \cdot 4!} \frac{\mathrm{d}^2}{\mathrm{d}x^2} (8 \cdot 7x^6 - 4 \cdot 6 \cdot 5x^4 + 6 \cdot 4 \cdot 3x^2 - 4 \cdot 2) \\
&= \frac{1}{2^4 \cdot 4!} \frac{\mathrm{d}}{\mathrm{d}x} (8 \cdot 7 \cdot 6x^5 - 4 \cdot 6 \cdot 5 \cdot 4x^3 + 6 \cdot 4 \cdot 3 \cdot 2x) \\
&= \frac{1}{2^4 \cdot 4!} (8 \cdot 7 \cdot 6 \cdot 5x^4 - 4 \cdot 6 \cdot 5 \cdot 4 \cdot 3x^2 + 6 \cdot 4 \cdot 3 \cdot 2) \\
&= \frac{1}{8} (35x^4 - 30x^2 + 3)
\end{aligned}$$

(Solution 2)

$$P_4(x) = \frac{1}{2^4 \cdot 4!} \frac{\mathrm{d}^4}{\mathrm{d}x^4} (x^2 - 1)^4$$

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\begin{aligned}
&= \frac{1}{2^4 \cdot 4!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(4(x^2-1)^3(2x)) \\
&= \frac{1}{2 \cdot 4!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x(x^2-1)^3) \\
&= \frac{1}{2 \cdot 4!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}((x^2-1)^3 + x \cdot 3(x^2-1)^2(2x)) \\
&= \frac{1}{2 \cdot 4!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}((x^2-1)^3 + 6x^2(x^2-1)^2) \\
&= \frac{1}{2 \cdot 4!}\frac{\mathrm{d}}{\mathrm{d}x}(3(x^2-1)^2(2x) + 12x(x^2-1)^2 + 6x^2 \cdot 2(x^2-1)(2x)) \\
&= \frac{6}{2 \cdot 4!}\frac{\mathrm{d}}{\mathrm{d}x}(x(x^2-1)^2 + 2x(x^2-1)^2 + 4x^3(x^2-1)) \\
&= \frac{1}{8}((x^2-1)^2 + x \cdot 2(x^2-1)(2x) \\
&\quad +2(x^2-1)^2 + 2x \cdot 2(x^2-1)(2x) + 12x^2(x^2-1) + 4x^3(2x)) \\
&= \frac{1}{8}(35x^4 - 30x^2 + 3)
\end{aligned}
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