Exercise 13-1

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Exercise The Fourier series $f(x) = x$ on the range $[-\pi, \pi]$ is

$$\sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx. \tag{1}$$

Rewrite this series in the form of a linear combination of complex exponential functions $\{e^{ikx} | k \in \mathbb{Z}\}$.

Solution 1 By the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2}$$

the following equation holds.

$$\begin{aligned} e^{-i\theta} &= e^{i(-\theta)} \\ &= \cos(-\theta) + i\sin(-\theta) \\ &= \cos\theta - i\sin\theta \end{aligned} \tag{3}$$

By adding the equations (2) and (3) we obtain the following equation.

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

By subtracting the equation (3) from (2) we obtain the following equation.

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

So we obtain the following equations.

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned} \tag{4}$$

By setting $\theta = kx$ in (4) we obtain

$$\sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}$$

By substituting RHS of this equation for LHS of this equation in (1) we obtain

$$\sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \frac{e^{ikx} - e^{-ikx}}{2i}.$$

We rewrite this series as follows.

$$\begin{aligned}
\sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \frac{e^{ikx} - e^{-ikx}}{2i} &= \sum_{k=1}^{\infty} -\frac{1}{k}(-1)^k \frac{e^{ikx} - e^{-ikx}}{i} \\
&= \sum_{k=1}^{\infty} -\frac{1}{k}(-1)^k (-i)(e^{ikx} - e^{-ikx}) \\
&= \sum_{k=1}^{\infty} \frac{1}{k}(-1)^k i(e^{ikx} - e^{-ikx}) \\
&= \sum_{k=1}^{\infty} \left\{ \frac{1}{k}(-1)^k i e^{ikx} - \frac{1}{k}(-1)^k i e^{-ikx} \right\} \\
&= \sum_{k=1}^{\infty} \left\{ \frac{1}{k}(-1)^k i e^{ikx} - \frac{1}{k}(-1)^k i e^{i(-k)x} \right\} \\
&= \sum_{k=1}^{\infty} \left\{ \frac{1}{k}(-1)^k i e^{ikx} + \frac{1}{-k}(-1)^k i e^{i(-k)x} \right\} \\
&= \sum_{k=1}^{\infty} \left\{ \frac{1}{k}(-1)^k i e^{ikx} + \frac{1}{-k}(-1)^{-k} i e^{i(-k)x} \right\} \\
&= \sum_{k=-\infty}^{\infty} c_k e^{ikx}
\end{aligned}$$

Here c_k is defined as follows.

$$c_k = \begin{cases} \frac{1}{k}(-1)^k i & k > 0 \\ 0 & k = 0 \\ \frac{1}{k}(-1)^k i & k < 0 \end{cases}$$

Note that $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ is defined as follows.

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx}$$

Solution 2 Here we calculate the series directly. Assume the following equation holds. (Note that there are no coefficients c_{-n}, ..., c_0, ..., c_n that satisfy the equation, but it's ok.)

$$f(x) = \sum_{l=-n}^{n} c_l e^{ilx}$$

Multiply e^{-ikx} to the both sides of this equation and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)e^{-ikx}\mathrm{d}x &= \int_{-\pi}^{\pi} e^{-ikx} \sum_{l=-n}^{n} c_l e^{ilx}\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \sum_{l=-n}^{n} c_l e^{ilx} e^{-ikx}\mathrm{d}x \\
&= \int_{-\pi}^{\pi} \sum_{l=-n}^{n} c_l e^{i(l-k)x}\mathrm{d}x \\
&= \sum_{l=-n}^{n} \int_{-\pi}^{\pi} c_l e^{i(l-k)x}\mathrm{d}x \\
&= \int_{-\pi}^{\pi} c_k e^{0}\mathrm{d}x \\
&= \int_{-\pi}^{\pi} c_k\mathrm{d}x \\
&= 2\pi c_k
\end{aligned}$$

So we obtain c_k as follows.

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\mathrm{d}x$$

When $k \neq 0$ we calculate c_k as follows.

$$\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-ikx}\mathrm{d}x \\
&= \frac{1}{2\pi}\left\{\left[x\frac{e^{-ikx}}{-ik}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{-ikx}}{-ik}\mathrm{d}x\right\} \\
&= \frac{1}{2\pi}\cdot\frac{\pi e^{-ik\pi} - (-\pi)e^{ik\pi}}{-ik} \\
&= \frac{1}{2\pi}\cdot\frac{\pi(-1)^k + \pi(-1)^k}{-ik} \\
&= \frac{1}{2\pi}\cdot\frac{2\pi(-1)^k}{-ik} \\
&= \frac{(-1)^k}{-ik} \\
&= \frac{1}{k}(-1)^k i
\end{aligned}$$

When $k = 0$ we calculate c_0 as follows.

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{0}\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\mathrm{d}x \\
&= 0
\end{aligned}$$

So we obtain the series

$$\sum_{k=-n}^{n} c_k e^{ikx}$$

where

$$c_k = \begin{cases} \frac{1}{k}(-1)^k i & k \neq 0 \\ 0 & k = 0. \end{cases}$$

The Fourier series is the limit of the above linear combination as n goes to infinity.

$$\lim_{n\to\infty}\sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

This is the same as the series obtained in Solution 1.

Comment Here we rewrite the series back to the series (1).

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} c_k e^{ikx} &= \lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx} \\
&= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} c_k e^{ikx} + \sum_{k=-1}^{-n} c_k e^{ikx} \right\} + c_0 e^0 \\
&= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} c_k e^{ikx} + \sum_{k=1}^{n} c_{-k} e^{i(-k)x} \right\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ c_k e^{ikx} + c_{-k} e^{i(-k)x} \right\} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ \frac{1}{k}(-1)^k i e^{ikx} + \frac{1}{-k}(-1)^{-k} i e^{i(-k)x} \right\} \\
&= \sum_{k=1}^{\infty} \left\{ \frac{1}{k}(-1)^k i(\cos kx + i \sin kx) \right. \\
&\qquad \left. - \frac{1}{k}(-1)^k i(\cos kx - i \sin kx) \right\} \\
&= \sum_{k=1}^{\infty} \left\{ \frac{1}{k}(-1)^k i \cos kx - \frac{1}{k}(-1)^k \sin kx \right. \\
&\qquad \left. - \frac{1}{k}(-1)^k i \cos kx - \frac{1}{k}(-1)^k \sin kx \right\} \\
&= \sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx
\end{aligned}$$

This is the series (1).