Classification of quadratic functions of two variables

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Caution This material concerns topics that are out of the scope of this class. This is for only those who are interested in classification of quadratic functions of two variables. It is required for you to have learned eigenvalues, eigenvectors, and diagonalizing matrices by orthogonal matrices.

1 Classification of J in Exercise 1

The function J for measuring the distance between the line and the points in Exercise 1

$$J = \frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2$$

is the following quadratic function of two variables a and b (cf. Solution 2 in Exercise 1).

$$J = \frac{1}{2}\{5a^2 + 3b^2 + 6ab + 8a + 2b + 5\}$$

As I mentioned in the lecture, quadratic functions of two variables are classified into elliptic, parabolic, and hyperbolic types. In this material, we show the function J is elliptic. We let the part other than the $\frac{1}{2}$ in J be $f(a, b)$.

$$f(a, b) = 5a^2 + 3b^2 + 6ab + 8a + 2b + 5$$

The subexpression $5a^2 + 3b^2 + 6ab$, which is called to have a *quadratic form*, can be represented by using matrices and vectors as follows.

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The subexpression $8a+2b$ can be represented by using row and column vectors as follows.

$$\begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus the function $f(a, b)$ can be represented as follows.

$$f(a, b) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + 5$$

We let the matrix $\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$ be A. We have selected A to be a symmetric matrix. Quadratic forms of a and b (terms of a^2, b^2, and ab) can always be represented by using symmetric matrices as in the above. The above expression can be rewritten by using A as follows.

$$f(a, b) = \begin{pmatrix} a & b \end{pmatrix} A \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + 5$$

We then diagonalize the matrix A by using orthogonal matrices. For that purpose we calculate x, y, and λ that satisfies the equation

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

and the inequation

$$\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equation $A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ can be represented by using the identity matrix I as follows.

$$(\lambda I - A) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since

$$|\lambda I - A| = 0$$

the following equation holds.

$$(\lambda - 5)(\lambda - 3) - 9 = 0$$

By solving this we obtain

$$\lambda = 4 \pm \sqrt{10}.$$

When $\lambda = 4 + \sqrt{10}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 + \sqrt{10} \end{pmatrix}$$

is obtained as a solution and by normalizing this vector we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{20 - 2\sqrt{10}}} \begin{pmatrix} 3 \\ -1 + \sqrt{10} \end{pmatrix}.$$

When $\lambda = 4 - \sqrt{10}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 - \sqrt{10} \end{pmatrix}$$

is obtained as a solution and by normalizing this vector we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{20 + 2\sqrt{10}}} \begin{pmatrix} 3 \\ -1 - \sqrt{10} \end{pmatrix}.$$

We make a matrix consisting of the above two unit vectors as follows.

$$U = \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} & \frac{3}{\sqrt{20+2\sqrt{10}}} \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} & \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} \end{pmatrix}$$

Note that this matrix U is a orthogonal matrix. By using the matrix U, the following equation holds.

$$A = U \begin{pmatrix} 4+\sqrt{10} & 0 \\ 0 & 4-\sqrt{10} \end{pmatrix} U^{\mathrm{T}}$$

Here we write the transposed matrix of U as U^{T}. By letting $\lambda_1 = 4+\sqrt{10}$ and $\lambda_2 = 4-\sqrt{10}$ the above equation can be rewritten as follows.

$$A = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{\mathrm{T}}$$

Note that the following equation also holds.

$$U^{\mathrm{T}} A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

By letting

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = U^{\mathrm{T}} \begin{pmatrix} a \\ b \end{pmatrix},$$

since U is a orthogonal matrix, the following equation holds.

$$U \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus the quadratic form $5a^2 + 3b^2 + 6ab$ can be rewritten as follows.

$$\begin{aligned} 5a^2 + 3b^2 + 6ab &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} a & b \end{pmatrix} A \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \left(\begin{pmatrix} a \\ b \end{pmatrix}, A \begin{pmatrix} a \\ b \end{pmatrix} \right) \end{aligned}$$

(obtained by representing a product of row and column vectors by an inner product)

$$\begin{aligned} &= \left(U \begin{pmatrix} a' \\ b' \end{pmatrix}, AU \begin{pmatrix} a' \\ b' \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} a' \\ b' \end{pmatrix}, U^{\mathrm{T}} A U \begin{pmatrix} a' \\ b' \end{pmatrix} \right) \end{aligned}$$

(obtained by the equality $(Ax, y) = (x, A^{\mathrm{T}} y)$)

$$
\begin{aligned}
&= \left(\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \lambda_1 a' \\ \lambda_2 b' \end{pmatrix} \right) \\
&= \lambda_1 a'^2 + \lambda_2 b'^2
\end{aligned}
$$

We call the obtained expression $\lambda_1 a'^2 + \lambda_2 b'^2$ the normal form of the quadratic form $5a^2 + 3b^2 + 6ab$.

The subexpression $8a + 2b$ can be rewritten as follows.

$$
\begin{aligned}
8a + 2b &= \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \begin{pmatrix} 8 & 2 \end{pmatrix} U \begin{pmatrix} a' \\ b' \end{pmatrix} \\
&= \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} & \frac{3}{\sqrt{20+2\sqrt{10}}} \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} & \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} \\
&= \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} a' + \frac{3}{\sqrt{20+2\sqrt{10}}} b' \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b' \end{pmatrix} \\
&= \frac{24}{\sqrt{20-2\sqrt{10}}} a' + \frac{24}{\sqrt{20+2\sqrt{10}}} b' + \frac{-2+2\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{-2-2\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b' \\
&= \frac{22+2\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{22-2\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b'
\end{aligned}
$$

So the function $f(a, b)$ can be rewritten as follows.

$$
f(a, b) = \lambda_1 a'^2 + \lambda_2 b'^2 + \frac{22+2\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{22-2\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b' + 5
$$

By completing the squares with respect to a' and b' we obtain

$$
f(a, b) = \lambda_1 \left(a' + \frac{11+\sqrt{10}}{\lambda_1 \sqrt{20-2\sqrt{10}}} \right)^2 + \lambda_2 \left(b' + \frac{11-\sqrt{10}}{\lambda_2 \sqrt{20+2\sqrt{10}}} \right)^2 + \frac{1}{6}.
$$

Since $J = \dfrac{1}{2} f(a, b)$ we obtain

$$
J = \frac{\lambda_1}{2} \left(a' + \frac{11+\sqrt{10}}{\lambda_1 \sqrt{20-2\sqrt{10}}} \right)^2 + \frac{\lambda_2}{2} \left(b' + \frac{11-\sqrt{10}}{\lambda_2 \sqrt{20+2\sqrt{10}}} \right)^2 + \frac{1}{12}.
$$

Since $\lambda_1 > 0$ and $\lambda_2 > 0$, $\dfrac{\lambda_1}{2} > 0$ and $\dfrac{\lambda_2}{2} > 0$ hold and thus the function J is elliptic. So when

$$
\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -\frac{11+\sqrt{10}}{\lambda_1 \sqrt{20-2\sqrt{10}}} \\ -\frac{11-\sqrt{10}}{\lambda_2 \sqrt{20+2\sqrt{10}}} \end{pmatrix}
$$

J takes a minimum value of $\dfrac{1}{12}$. Since $\begin{pmatrix} a \\ b \end{pmatrix} = U \begin{pmatrix} a' \\ b' \end{pmatrix}$ we obtain

$$\begin{aligned}
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} & \frac{3}{\sqrt{20+2\sqrt{10}}} \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} & \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} \end{pmatrix} \begin{pmatrix} -\frac{11+\sqrt{10}}{\lambda_1\sqrt{20-2\sqrt{10}}} \\ -\frac{11-\sqrt{10}}{\lambda_2\sqrt{20+2\sqrt{10}}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{2} \\ \frac{7}{6} \end{pmatrix}.
\end{aligned}$$

So when $a = -\dfrac{3}{2}$ and $b = \dfrac{7}{6}$ the function J takes a minimum value of $\dfrac{1}{12}$.

Note The function J is obtained by linear transforming the elliptic quadratic function of two variables

$$\frac{\lambda_1}{2}a^2 + \frac{\lambda_2}{2}b^2$$

by using the matrix U^{T}, which represents a rotation, and parallel shifting $-\dfrac{3}{2}$ in the a-axis, $\dfrac{7}{6}$ in the b-axis, and $\dfrac{1}{12}$ in the J-axis.

2 Classification of quadratic two variable functions

Here we classify quadratic two variable functions of the following form.

$$f(x, y) = ax^2 + by^2 + cxy + dx + ey + f$$

The subexpression $ax^2 + by^2 + cxy$, which is called to have a *quadratic form*, can be represented by using matrices and vectors as follows.

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}c \\ \frac{1}{2}c & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The subexpression $dx + ey$ can be represented by using row and column vectors as follows.

$$\begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus the function $f(x, y)$ can be represented as follows.

$$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}c \\ \frac{1}{2}c & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f$$

We let the matrix $\begin{pmatrix} a & \frac{1}{2}c \\ \frac{1}{2}c & b \end{pmatrix}$ be A. We have selected A to be a symmetric matrix. Quadratic forms of x and y (terms of x^2, y^2, and xy) can always be represented by using symmetric matrices as in the above. The above expression can be rewritten by using A as follows.

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f$$

We then diagonalize the matrix A by using orthogonal matrices. For that purpose we calculate x, y, and λ that satisfies the equation

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

and the inequation

$$\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equation $A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ can be represented by using the identity matrix I as follows.

$$(\lambda I - A) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since

$$|\lambda I - A| = 0$$

the following equation holds.

$$(\lambda - a)(\lambda - b) - \frac{1}{4}c^2 = 0$$

By solving this we obtain

$$\lambda = \frac{a + b \pm \sqrt{(a-b)^2 + c^2}}{2}.$$

We let

$$\lambda_1 = \frac{a + b + \sqrt{(a-b)^2 + c^2}}{2}$$
$$\lambda_2 = \frac{a + b - \sqrt{(a-b)^2 + c^2}}{2}.$$

As for λ_1,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c \\ \lambda_1 - a \end{pmatrix}$$

is obtained as a solution and by normalizing this vector we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{\frac{1}{4}c^2 + (\lambda_1 - a)^2}} \begin{pmatrix} \frac{1}{2}c \\ \lambda_1 - a \end{pmatrix}.$$

As for λ_2,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}c \\ \lambda_2 - a \end{pmatrix}$$

is obtained as a solution and by normalizing this vector we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{\frac{1}{4}c^2 + (\lambda_2 - a)^2}} \begin{pmatrix} \frac{1}{2}c \\ \lambda_2 - a \end{pmatrix}.$$

We make a matrix consisting of the above two unit vectors as follows.

$$U = \begin{pmatrix} \frac{\frac{1}{2}c}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}} & \frac{\frac{1}{2}c}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}} \\ \frac{\lambda_1-a}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}} & \frac{\lambda_2-a}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}} \end{pmatrix}$$

Note that this matrix U is a orthogonal matrix. By using the matrix U, the following equation holds.

$$A = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{\mathrm{T}}$$

Here we write the transposed matrix of U as U^{T}. Note that the following equation also holds.

$$U^{\mathrm{T}}AU = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

By letting

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = U^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix},$$

since U is a orthogonal matrix, the following equation holds.

$$U \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Thus the quadratic form $ax^2 + by^2 + cxy$ can be rewritten as follows.

$$\begin{aligned} ax^2 + by^2 + cxy &= \begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \left(\begin{pmatrix} x \\ y \end{pmatrix}, A \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &\quad \text{(obtained by representing a product of row and column vectors by an inner product)} \\ &= \left(U \begin{pmatrix} x' \\ y' \end{pmatrix}, AU \begin{pmatrix} x' \\ y' \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} x' \\ y' \end{pmatrix}, U^{\mathrm{T}}AU \begin{pmatrix} x' \\ y' \end{pmatrix} \right) \\ &\quad \text{(obtained by the equality } (Ax, y) = (x, A^{\mathrm{T}}y)\text{)} \end{aligned}$$

$$
\begin{aligned}
&= \left(\begin{pmatrix} x' \\ y' \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} x' \\ y' \end{pmatrix}, \begin{pmatrix} \lambda_1 x' \\ \lambda_2 y' \end{pmatrix} \right) \\
&= \lambda_1 x'^2 + \lambda_2 y'^2
\end{aligned}
$$

We call the obtained expression $\lambda_1 x'^2 + \lambda_2 y'^2$ the normal form of the quadratic form $ax^2 + by^2 + cxy$.

The subexpression $dx + ey$ can be rewritten as follows.

$$
\begin{aligned}
dx + ey &= \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} d & e \end{pmatrix} U \begin{pmatrix} x' \\ y' \end{pmatrix} \\
&= \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} \frac{\frac{1}{2}c}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}} & \frac{\frac{1}{2}c}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}} \\ \frac{\lambda_1-a}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}} & \frac{\lambda_2-a}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\
&= \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} \frac{\frac{1}{2}c}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}}x' + \frac{\frac{1}{2}c}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}}y' \\ \frac{\lambda_1-a}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}}x' + \frac{\lambda_2-a}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}}y' \end{pmatrix} \\
&= \frac{\frac{1}{2}cd}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}}x' + \frac{\frac{1}{2}cd}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}}y' \\
&\quad + \frac{e(\lambda_1-a)}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}}x' + \frac{e(\lambda_2-a)}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}}y' \\
&= \frac{\frac{1}{2}cd + e(\lambda_1-a)}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}}x' + \frac{\frac{1}{2}cd + e(\lambda_2-a)}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}}y'
\end{aligned}
$$

So the function $f(x, y)$ can be rewritten as follows.

$$
f(x,y) = \lambda_1 x'^2 + \lambda_2 y'^2 + \frac{\frac{1}{2}cd + e(\lambda_1-a)}{\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}}x' + \frac{\frac{1}{2}cd + e(\lambda_2-a)}{\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}}y' + f
$$

By completing the squares with respect to x' and y' we obtain

$$
\begin{aligned}
&f(x,y) \\
&= \lambda_1 \left(x' + \frac{\frac{1}{2}cd + e(\lambda_1-a)}{2\lambda_1\sqrt{\frac{1}{4}c^2+(\lambda_1-a)^2}} \right)^2 + \lambda_2 \left(y' + \frac{\frac{1}{2}cd + e(\lambda_2-a)}{2\lambda_2\sqrt{\frac{1}{4}c^2+(\lambda_2-a)^2}} \right)^2 \\
&\quad - \frac{\left\{\frac{1}{2}cd + e(\lambda_1-a)\right\}^2}{4\lambda_1\left\{\frac{1}{4}c^2+(\lambda_1-a)^2\right\}} - \frac{\left\{\frac{1}{2}cd + e(\lambda_2-a)\right\}^2}{4\lambda_2\left\{\frac{1}{4}c^2+(\lambda_2-a)^2\right\}} + f
\end{aligned}
$$

The function $f(x,y)$ is called *elliptic* when $\lambda_1 > 0$ and $\lambda_2 > 0$ or $\lambda_1 < 0$ and $\lambda_2 < 0$, *paratolic* when $\lambda_1 \neq 0$ and $\lambda_2 = 0$ or $\lambda_1 = 0$ and $\lambda_2 \neq 0$, and *hyperbolic* when $\lambda_1 > 0$ and $\lambda_2 < 0$ or $\lambda_1 < 0$ and $\lambda_2 > 0$.

In general, λ_1 and λ_2 may be positive, negative, or zero. In the approximation problems by a linear function, $\lambda_1 > 0$ and $\lambda_2 > 0$ usually, and in special cases, $\lambda_1 > 0$ and $\lambda_2 = 0$, which we show in the next section. That is, J is usually elliptic and in special cases parabolic, and never hyperbolic, in the approximation problems by a linear function.

3 J in the approximation problems by a linear function

Here we calculate the signs of λ_1 and λ_2 in the approximation problems by a lienar function. The function J for measuring the distance between the line $f(x) = ax + b$ and the points $(x_1, y_1), \ldots, (x_N, y_N)$ was

$$J(a, b) = \frac{1}{2}\sum_{i=1}^{N}\{f(x_i) - y_i\}^2.$$

By expanding this formula we obtain

$$J(a, b) = \frac{1}{2}a^2\sum_{i=1}^{N} x_i^2 + \frac{1}{2}b^2 N + ab\sum_{i=1}^{N} x_i - a\sum_{i=1}^{N} x_i y_i - b\sum_{i=1}^{N} y_i + \frac{1}{2}\sum_{i=1}^{N} y_i^2.$$

So λ in the previous section is obtained as follows.

$$\lambda = \frac{\frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{1}{2}N \pm \sqrt{\left(\frac{1}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}N\right)^2 + \left(\sum_{i=1}^{N} x_i\right)^2}}{2}$$

We let

$$\lambda_1 = \frac{\frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{1}{2}N + \sqrt{\left(\frac{1}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}N\right)^2 + \left(\sum_{i=1}^{N} x_i\right)^2}}{2}$$

$$\lambda_2 = \frac{\frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{1}{2}N - \sqrt{\left(\frac{1}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}N\right)^2 + \left(\sum_{i=1}^{N} x_i\right)^2}}{2}.$$

Firstly, $\lambda_1 > 0$. (We assume $N \geq 1$, that is, there is at least one point in the experiment.) We calculate the sign of the following expression in order to obtain the sign of λ_2.

$$\left\{\frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{1}{2}N\right\}^2 - \left\{\sqrt{\left(\frac{1}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}N\right)^2 + \left(\sum_{i=1}^{N} x_i\right)^2}\right\}^2$$

Note that the sign of this expression and the sign of λ_2 is the same. By tranfroming this expression, we obtain

$$\begin{aligned}
&\left\{\frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{1}{2}N\right\}^2 - \left\{\sqrt{\left(\frac{1}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}N\right)^2 + \left(\sum_{i=1}^{N} x_i\right)^2}\right\}^2 \\
&= \frac{1}{4}\left\{\sum_{i=1}^{N} x_i^2\right\}^2 + \frac{1}{4}N^2 + \frac{1}{2}N\sum_{i=1}^{N} x_i^2 \\
&\quad - \left\{\frac{1}{4}\left(\sum_{i=1}^{N} x_i^2\right)^2 + \frac{1}{4}N^2 - \frac{1}{2}N\sum_{i=1}^{N} x_i^2 + \left(\sum_{i=1}^{N} x_i\right)^2\right\} \\
&= N\sum_{i=1}^{N} x_i^2 - \left\{\sum_{i=1}^{N} x_i\right\}^2 .
\end{aligned}$$

This expression is 0 when $N = 1$. When $N \geq 2$ this expression is equal to the following expression.

$$\sum_{i<j}(x_i - x_j)^2$$

Note that $\sum_{i<j}$ represents summing up all the expressions where $i < j$. So when

$$x_1 = \ldots = x_N,$$

that is, when all the values of x-coordinate are equal, the expression is 0, and positive otherwise. Such situations do not occur in the actual experiments. So in usual cases λ_1 and λ_2 are both positive and $J(a, b)$ is elliptic. That is, $J(a, b)$ takes a minimum value for only one point (a, b). When all the values of x-coordinate are equal, $\lambda_1 > 0$ and $\lambda_2 = 0$, so $J(a, b)$ is parabolic. That is, $J(a, b)$ takes a minimum value for infinitely many points (a, b).