Supplement 9:
A derivation of a step in the proof of Triangle inequality

Isao Sasano

This material shows the following step in the proof of Triangle inequality by using the axioms for inner product.

$$(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{u}) + 2(\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{v},\boldsymbol{v})$$

The axioms for inner product are as follows. Let the inner procudt space be $\mathcal{L}$.

1. Positive definiteness:
   $\forall \boldsymbol{u} \in \mathcal{L}.\ (u,u) \geq 0$. The equality $(\boldsymbol{u},\boldsymbol{u}) = 0$ holds iff $\boldsymbol{u} = \boldsymbol{0}$.
2. Symmetry:
   $\forall \boldsymbol{u},\boldsymbol{v} \in \mathcal{L}.\ (\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{v},\boldsymbol{u})$.
3. Linearity:
   $\forall c_1, c_2 \in \mathbb{R} \wedge \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v} \in \mathcal{L}.\ (c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2, \boldsymbol{v}) = c_1(\boldsymbol{u}_1,\boldsymbol{v}) + c_2(\boldsymbol{u}_2,\boldsymbol{v})$.

We derive RHS from LHS as follows.

$$\begin{array}{cl}
 & \underline{(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v})} \\
= & \quad \{\text{ linearity }\} \\
 & \underline{(\boldsymbol{u},\boldsymbol{u}+\boldsymbol{v})} + \underline{(\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v})} \\
= & \quad \{\text{ symmetry }\} \\
 & (\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u}) + (\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}) \\
= & \quad \{\text{ linearity }\} \\
 & (\boldsymbol{u},\boldsymbol{u}) + \underline{(\boldsymbol{v},\boldsymbol{u})} + (\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{v},\boldsymbol{v}) \\
= & \quad \{\text{ symmetry }\} \\
 & (\boldsymbol{u},\boldsymbol{u}) + \underline{(\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{u},\boldsymbol{v})} + (\boldsymbol{v},\boldsymbol{v}) \\
= & \quad \{\text{ arithmetic of real numbers }\} \\
 & (\boldsymbol{u},\boldsymbol{u}) + 2(\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{v},\boldsymbol{v})
\end{array}$$