Example 11

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Example Calculate the value of the following series by using the Parseval's equality for the Fourier series of $f(x) = x$ on the range $[-\pi, \pi]$ following the steps (1)-(5).

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

(1) Calculate the linear combination of the following orthogonal functions that is closest to the function $f(x)$. As for the measure of the distance, use (the half of) the integral of the square of the difference on the range $[-\pi, \pi]$.

$$\{\frac{1}{2}, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

(2) Obtain the Fourier series of $f(x)$ on the range $[-\pi, \pi]$. (The Fourier series of $f(x) = x$ is the limit of the linear combination obtained in (1) as n goes to infinity.)

(3) Normalise the series obtained in (2).

(4) Write down the Parseval's equality for the series obtained in (4).

(5) Calculate the value of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Note that in an inner product space $\mathcal{L}$, when the approaximation, in the sense of the least square, of $\boldsymbol{u} \in \mathcal{L}$ by a linear comination of an orthonormal basis $\{\boldsymbol{e}_i | i \geq 1\}$ in $\mathcal{L}$

$$\sum_{k=1}^{n} c_k \boldsymbol{e}_k$$

converges to $\boldsymbol{u}$ in the sense that the norm of the difference converges to 0 as n goes to infinity, the following equation, called Parseval's equality, holds.

$$\|\boldsymbol{u}\|^2 = \sum_{k=1}^{\infty} c_k^2$$

Solution

(1) Assume the following equation holds. (Note: There are no coefficients $a_0, \ldots, a_n, b_1, \ldots, b_n$ that satisfy the equation, but it's ok.)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \tag{1}$$

Integrate the both sides of the equation (1) on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}a_0\mathrm{d}x \\ &= a_0\pi \end{aligned}$$

Then we calculate a_0 as follows.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\mathrm{d}x \\ &= 0 \end{aligned}$$

Multiply the both sides of the equation (1) by $\cos kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x &= a_k\int_{-\pi}^{\pi} \cos^2 kx\mathrm{d}x \\ &= a_k\pi \end{aligned}$$

Then we calculate a_k as follows.

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx\mathrm{d}x \\ &= 0 \quad \text{(since } x\cos kx \text{ is an odd function)} \end{aligned}$$

Multiply the both sides of the equation (1) by $\sin kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin kx\mathrm{d}x &= b_k \int_{-\pi}^{\pi} \sin^2 kx\mathrm{d}x \\ &= b_k\pi\end{aligned}$$

Then we calculate b_k as follows.

$$\begin{aligned}b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x\end{aligned}$$

Here we calculate the integral $\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x$.

$$\begin{aligned}\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x &= \left[x\frac{-\cos kx}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-\cos kx}{k}\mathrm{d}x \\ &= -\frac{1}{k}\left[x\cos kx\right]_{-\pi}^{\pi} \qquad \left(\text{since } \int_{-\pi}^{\pi}\frac{-\cos kx}{k}\mathrm{d}x \text{ is } 0\right) \\ &= -\frac{1}{k}(\pi\cos\pi k - (-\pi)\cos(-\pi k)) \\ &= -\frac{1}{k}(\pi\cos\pi k + \pi\cos\pi k) \\ &= -\frac{2\pi}{k}\cos\pi k \\ &= -\frac{2\pi}{k}(-1)^k\end{aligned}$$

We resume the calculation of b_k.

$$\begin{aligned}b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x \\ &= \frac{1}{\pi}\cdot -\frac{2\pi}{k}(-1)^k \\ &= -\frac{2}{k}(-1)^k\end{aligned}$$

So the linear combination that is closest to the function $f(x)$ is

$$\sum_{k=1}^{n} -\frac{2}{k}(-1)^k \sin kx.$$

(2) The Fourier expansion of $f(x) = x$ is the limit of the above linear combination as n goes to infinity:

$$\sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx.$$

(3) Firstly we calculate the norm of $\sin kx$.

$$\begin{aligned} \|\sin kx\| &= \sqrt{(\sin kx, \sin kx)} \\ &= \sqrt{\int_{\pi}^{\pi} \sin^2 kx \mathrm{d}x} \\ &= \sqrt{\pi} \end{aligned}$$

So the Fourier series of $f(x) = x$ is normalized as follows.

$$\sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx = \sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sqrt{\pi} \cdot \frac{\sin kx}{\sqrt{\pi}}$$

(4) By the Parseval's equality, we obtain the following equation.

$$\|f\|^2 = \sum_{k=1}^{\infty} \left(-\frac{2}{k}(-1)^k \sqrt{\pi} \right)^2 \tag{2}$$

The left hand side of the equation (2) is calculated as follows.

$$\begin{aligned} \|f\|^2 &= (f, f) \\ &= \int_{-\pi}^{\pi} f(x)^2 \mathrm{d}x \\ &= \int_{-\pi}^{\pi} x^2 \mathrm{d}x \\ &= \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} \\ &= \left(\frac{\pi^3}{3} - (-\frac{\pi^3}{3}) \right) \\ &= \frac{2}{3}\pi^3 \end{aligned}$$

The right hand side of the equation (2) is calculated as follows.

$$\begin{aligned}
\text{RHS} &= \sum_{k=1}^{\infty}\left(-\frac{2}{k}(-1)^k\sqrt{\pi}\right)^2 \\
&= \sum_{k=1}^{\infty}\frac{4\pi}{k^2}((-1)^k)^2 \\
&= \sum_{k=1}^{\infty}\frac{4\pi}{k^2}(-1)^{2k} \\
&= \sum_{k=1}^{\infty}\frac{4\pi}{k^2}((-1)^2)^k \\
&= \sum_{k=1}^{\infty}\frac{4\pi}{k^2}1^k \\
&= \sum_{k=1}^{\infty}\frac{4\pi}{k^2} \\
&= 4\pi\sum_{k=1}^{\infty}\frac{1}{k^2}
\end{aligned}$$

So the Parceval's equality for the Fourier series of $f(x) = x$ is obtained as follows.

$$\frac{2}{3}\pi^3 = 4\pi\sum_{k=1}^{\infty}\frac{1}{k^2}$$

(5) So the value of the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is obtained as follows.

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{4\pi}\cdot\frac{2}{3}\pi^3 = \frac{1}{6}\pi^2$$

(Note) The obtained value $\frac{1}{6}\pi^2$ is the value of the zeta function $\zeta(n)$ when $n = 2$. The zeta function is given as follows.

$$\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}$$

So $\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{6}\pi^2$.