Exercise 14-1

Isao Sasano

Exercise Calculate the Fourier transform of the following function $f(x)$.

$$f(x) = \begin{cases} x & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

Here we define the Fourier transform and the inverse Fourier transform as follows[1].

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\mathrm{d}\omega \\ F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x \end{aligned}$$

The above equations are definitions of transformations. Note that the inverse Fourier transform of the Fourier transform of $f(x)$ is not necessarily equal to $f(x)$.

[1] In the definitions the constant factor depends on textbooks.

Solution

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x \\
&= \int_0^L xe^{-i\omega x}\mathrm{d}x \\
&= \left[x\frac{e^{-i\omega x}}{-i\omega}\right]_0^L - \int_0^L \frac{e^{-i\omega x}}{-i\omega}\mathrm{d}x \\
&= \frac{Le^{-i\omega L}}{-i\omega} - \left[\frac{e^{-i\omega x}}{-\omega^2}\right]_0^L \\
&= \frac{Le^{-i\omega L}}{-i\omega} + \left[\frac{e^{-i\omega x}}{\omega^2}\right]_0^L \\
&= i\frac{Le^{-i\omega L}}{\omega} + \frac{e^{-i\omega L}-1}{\omega^2} \\
&= i\frac{\omega Le^{-i\omega L}}{\omega^2} + \frac{e^{-i\omega L}-1}{\omega^2} \\
&= \frac{i\omega Le^{-i\omega L} + e^{-i\omega L} - 1}{\omega^2} \\
&= \frac{e^{-i\omega L}(1+i\omega L) - 1}{\omega^2} \\
&= \frac{e^{-iL\omega}(1+iL\omega) - 1}{\omega^2}
\end{aligned}$$

Comment The inverse Fourier transform of $F(w)$ obtained above is as follows.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-iL\omega}(1+iL\omega)-1}{\omega^2}e^{i\omega x}\mathrm{d}\omega = \begin{cases} x & 0 < x < L \\ L/2 & x = L \\ 0 & \text{otherwise} \end{cases}$$

Note that I do not obtain the inverse Fourier transform by calculating the above integral but obtained it from $f(x)$ by the following equation (Refer to Theorem 1 in the previous document).

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \frac{f(x+0)+f(x-0)}{2}$$