Exercise 11

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Exercise Calculate the value of the following series by using the Parseval's equality for the Fourier series of $f(x) = x^2$ on the range $[-\pi, \pi]$ following the steps (1)-(5).

$$\sum_{k=1}^{\infty} \frac{1}{k^4}$$

(1) Calculate the linear combination of the following orthogonal functions that is closest to the function $f(x)$. As for the measure of the distance, use (the half of) the integral of the square of the difference on the range $[-\pi, \pi]$.

$$\{\frac{1}{2}, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx\}$$

(2) Obtain the Fourier series of $f(x)$ on the range $[-\pi, \pi]$. (The Fourier series of $f(x) = x^2$ is the limit of the linear combination obtained in (1) as n goes to infinity.)

(3) Normalise the series obtained in (2).

(4) Write down the Parseval's equality for the series obtained in (3).

(5) Calculate the value of the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$.

Note that in an inner product space $\mathcal{L}$, when the approaximation, in the sense of the least square, of $\boldsymbol{u} \in \mathcal{L}$ by a linear comination of an orthonormal basis $\{\boldsymbol{e}_i | i \geq 1\}$ in $\mathcal{L}$

$$\sum_{k=1}^{n} c_k \boldsymbol{e}_k$$

converges to $\boldsymbol{u}$ in the sense that the norm of the difference converges to 0 as n goes to infinity, the following equation, called Parseval's equality, holds.

$$\|\boldsymbol{u}\|^2 = \sum_{k=1}^{\infty} c_k^2$$

Solution

(1) Assume the following equation holds. (Note: There are no coefficients a_0, ..., a_n, b_1, ..., b_n that satisfy the equation, but it's ok.)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \tag{1}$$

Integrate the both sides of the equation (1) on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\mathrm{d}x &= \int_{-\pi}^{\pi} \frac{1}{2}a_0\mathrm{d}x \\ &= a_0\pi \end{aligned}$$

Then we calculate a_0 as follows.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x \\ &= \frac{2}{3}\pi^2 \end{aligned}$$

Multiply the both sides of the equation (1) by $\cos kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x &= a_k\int_{-\pi}^{\pi} \cos^2 kx\mathrm{d}x \\ &= a_k\pi \end{aligned}$$

Then we calculate a_k as follows.

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos kx\mathrm{d}x \\ &= \frac{1}{\pi}\left\{\left[x^2\frac{\sin kx}{k}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\frac{\sin kx}{k}\mathrm{d}x\right\} \\ &= -\frac{2}{\pi k}\int_{-\pi}^{\pi} x\sin kx\mathrm{d}x \qquad \left(\text{since } \left[x^2\frac{\sin kx}{k}\right]_{-\pi}^{\pi} \text{ is } 0\right) \end{aligned}$$

Here we calculate the integral $\int_{-\pi}^{\pi} x \sin kx \mathrm{d}x$.

$$\begin{aligned}
\int_{-\pi}^{\pi} x \sin kx \mathrm{d}x &= \left[x \frac{-\cos kx}{k} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-\cos kx}{k} \mathrm{d}x \\
&= -\frac{1}{k} \left[x \cos kx \right]_{-\pi}^{\pi} \qquad \left(\text{since } \int_{-\pi}^{\pi} \frac{-\cos kx}{k} \mathrm{d}x \text{ is } 0\right) \\
&= -\frac{1}{k} (\pi \cos \pi k - (-\pi) \cos(-\pi k)) \\
&= -\frac{1}{k} (\pi \cos \pi k + \pi \cos \pi k) \\
&= -\frac{2\pi}{k} \cos \pi k \\
&= -\frac{2\pi}{k} (-1)^k
\end{aligned}$$

We resume the calculation of a_k.

$$\begin{aligned}
a_k &= -\frac{2}{\pi k} \left(-\frac{2\pi}{k} (-1)^k\right) \\
&= \frac{4}{k^2} (-1)^k
\end{aligned}$$

Multiply the both sides of the equation (1) by $\sin kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x) \sin kx \mathrm{d}x &= b_k \int_{-\pi}^{\pi} \sin^2 kx \mathrm{d}x \\
&= b_k \pi
\end{aligned}$$

Then we calculate b_k as follows.

$$\begin{aligned}
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \mathrm{d}x \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin kx \mathrm{d}x \\
&= 0 \quad (\text{since } x^2 \sin kx \text{ is an odd function})
\end{aligned}$$

So the linear combination that is closest to the function $f(x)$ is

$$\frac{2}{3}\pi^2 \cdot \frac{1}{2} + \sum_{k=1}^{n} \frac{4}{k^2} (-1)^k \cos kx.$$

(2) The Fourier expansion of $f(x) = x^2$ is the limit of the above linear combination as n goes to infinity:

$$\frac{2}{3}\pi^2 \cdot \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \cos kx.$$

(3) Firstly we calculate the norm of $\frac{1}{2}$ and $\cos kx$.

$$\begin{aligned}
\|\frac{1}{2}\| &= \sqrt{\left(\frac{1}{2}, \frac{1}{2}\right)} \\
&= \sqrt{\int_{-\pi}^{\pi} \left(\frac{1}{2}\right)^2 \mathrm{d}x} \\
&= \sqrt{\int_{-\pi}^{\pi} \frac{1}{4} \mathrm{d}x} \\
&= \sqrt{\left[\frac{1}{4}x\right]_{-\pi}^{\pi}} \\
&= \sqrt{\frac{\pi}{2}} \\
\|\cos kx\| &= \sqrt{(\cos kx, \cos kx)} \\
&= \sqrt{\int_{\pi}^{\pi} \cos^2 kx \mathrm{d}x} \\
&= \sqrt{\pi}
\end{aligned}$$

So the Fourier series of $f(x) = x^2$ is normalized as follows.

$$\begin{aligned}
&\frac{2}{3}\pi^2 \cdot \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \cos kx \\
&\quad = \frac{2}{3}\pi^2 \sqrt{\frac{\pi}{2}} \cdot \frac{\frac{1}{2}}{\sqrt{\frac{\pi}{2}}} + \sum_{k=1}^{\infty} \frac{4}{k^2}(-1)^k \sqrt{\pi} \cdot \frac{\cos kx}{\sqrt{\pi}}
\end{aligned}$$

(4) By the Parseval's equality, we obtain the following equation.

$$\|f\|^2 = \left(\frac{2}{3}\pi^2 \sqrt{\frac{\pi}{2}}\right)^2 + \sum_{k=1}^{\infty} \left(\frac{4}{k^2}(-1)^k \sqrt{\pi}\right)^2 \tag{2}$$

The left hand side of the equation (2) is calculated as follows.

$$\begin{aligned}
\|f\|^2 &= (f, f) \\
&= \int_{-\pi}^{\pi} f(x)^2 \mathrm{d}x \\
&= \int_{-\pi}^{\pi} x^4 \mathrm{d}x \\
&= \left[\frac{x^5}{5}\right]_{-\pi}^{\pi} \\
&= \left(\frac{\pi^5}{5} - (-\frac{\pi^5}{5})\right) \\
&= \frac{2}{5}\pi^5
\end{aligned}$$

The right hand side of the equation (2) is calculated as follows.

$$\begin{aligned}
\text{RHS} &= \left(\frac{2}{3}\pi^2\sqrt{\frac{\pi}{2}}\right)^2 + \sum_{k=1}^{\infty}\left(\frac{4}{k^2}(-1)^k\sqrt{\pi}\right)^2 \\
&= \frac{4}{9}\pi^4 \cdot \frac{\pi}{2} + \sum_{k=1}^{\infty}\frac{16}{k^4}\pi \\
&= \frac{2}{9}\pi^5 + 16\pi\sum_{k=1}^{\infty}\frac{1}{k^4}
\end{aligned}$$

So the Parceval's equality for the Fourier series of $f(x) = x^2$ is obtained as follows.

$$\frac{2}{5}\pi^5 = \frac{2}{9}\pi^5 + 16\pi\sum_{k=1}^{\infty}\frac{1}{k^4}$$

(5) We calculate the value of the series $\sum_{k=1}^{\infty}\frac{1}{k^4}$.

$$\begin{aligned}
16\pi\sum_{k=1}^{\infty}\frac{1}{k^4} &= \frac{2}{5}\pi^5 - \frac{2}{9}\pi^5 \\
&= \frac{18}{45}\pi^5 - \frac{10}{45}\pi^5 \\
&= \frac{8}{45}\pi^5
\end{aligned}$$

So the value of the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is obtained as follows.

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^4} &= \frac{1}{16\pi} \cdot \frac{8}{45}\pi^5 \\ &= \frac{1}{90}\pi^4\end{aligned}$$

(Note) The obtained value $\frac{1}{90}\pi^4$ is the value of the zeta function $\zeta(n)$ when $n = 4$. The zeta function is given as follows.

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

So $\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{90}\pi^4$.