Solutions for Mid-term examination

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Problem 1 (10 points) Fit a straight line (a linear function) to the three points $(0,0),(1,1),(3,4)$ so that the half of the sum of the squares of the distances of those points from the straight line is minimum, where the distance is measured in the vertical direction (the y-direction).

Solution Let the line (the linear function) be $f(x) = ax+b$ and $(x_1, y_1) = (0,0), (x_2, y_2) = (1,1), (x_3, y_3) = (3,4)$. The half of the sum of the squares of the distances of these points from the line is given as follows.

$$J = \frac{1}{2}\sum_{i=1}^{3}(f(x_i) - y_i)^2 = \frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2$$

J takes the minimum value in the point where the partial derivatives of J with respect to a and b are 0.

$$\frac{\partial J}{\partial a} = 0, \frac{\partial J}{\partial b} = 0$$

Firstly the partial derivative of J with respect to a is calculated as follows.

$$\begin{aligned}
\frac{\partial J}{\partial a} &= \frac{\partial}{\partial a}\{\frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2\} \\
&= \frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial a}(ax_i + b - y_i)^2 \\
&= \frac{1}{2}\sum_{i=1}^{3}2(ax_i + b - y_i)x_i \\
&= \sum_{i=1}^{3}(ax_i + b - y_i)x_i \\
&= \sum_{i=1}^{3}(ax_i^2 + bx_i - x_i y_i) \\
&= a\sum_{i=1}^{3}x_i^2 + b\sum_{i=1}^{3}x_i - \sum_{i=1}^{3}x_i y_i
\end{aligned}$$

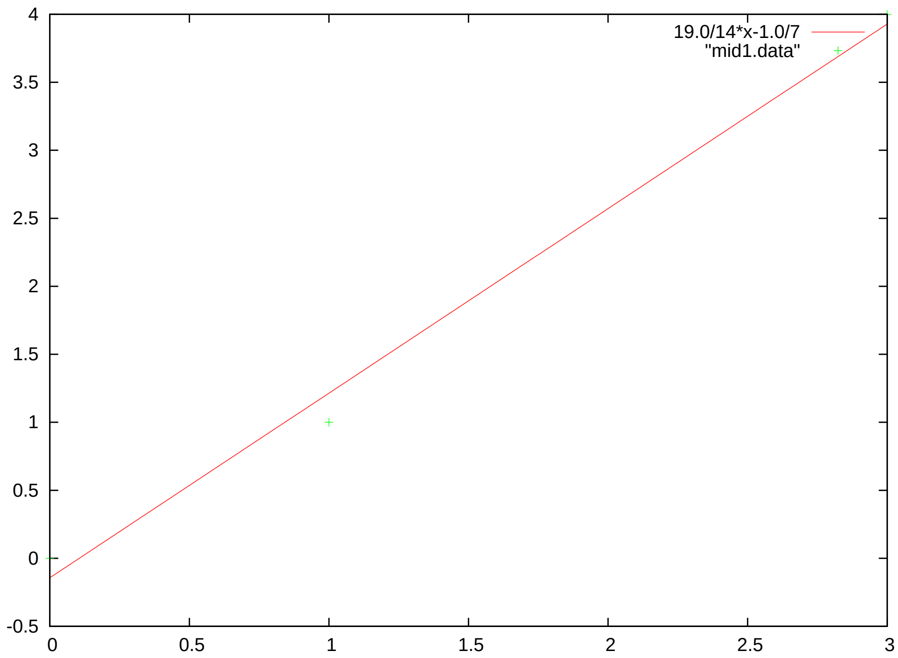

Figure 1: The straight line closest to the given three points

Secondly the partial derivative of J with respect to b is calculated as follows.

$$\begin{aligned}\frac{\partial J}{\partial b} &= \frac{\partial}{\partial b}\{\frac{1}{2}\sum_{i=1}^{3}(ax_i+b-y_i)^2\} \\ &= \frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial b}(ax_i+b-y_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{3}2(ax_i+b-y_i) \\ &= \sum_{i=1}^{3}(ax_i+b-y_i) \\ &= a\sum_{i=1}^{3}x_i+b\sum_{i=1}^{3}1-\sum_{i=1}^{3}y_i\end{aligned}$$

Then we obtain the system of equations

$$\begin{aligned}10a+4b-13 &= 0 \\ 4a+3b-5 &= 0\end{aligned}$$

and $a=\frac{19}{14}, b=-\frac{1}{7}$ is the solution. Hence the function is obtained as follows.

$$f(x)=\frac{19}{14}x-\frac{1}{7}$$

Supplement: The function is depicted with the three points in Fig. 1.

Problem 2 (10 points) Fit a parabola (a square function) to the four points $(-1,0),(0,-1),(1,0),(2,1)$ so that the half of the sum of the squares of the distances of those points from the parabola is minimum, where the distance is measured in the vertical direction (the y-direction).

Solution This problem is same as Exercise 2.

Problem 3 (10 points) Approximate a column vector $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ by a linear combination of the column vectors $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (i.e., $\sum_{k=1}^{2} c_k \boldsymbol{u}_k = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ for some c_1 and c_2). As for the measure of the distance, use the half of the square of the norm of the difference of $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2$ and $\boldsymbol{a}$: $J = \frac{1}{2}\left\| \sum_{k=1}^{2} c_k\boldsymbol{u}_k - \boldsymbol{a} \right\|^2$. The norm of a column vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is defined to be $\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x},\boldsymbol{x})} = \sqrt{\sum_{k=1}^{3} x_k^2}$. You may use the normal equation.

$$\begin{pmatrix} (\boldsymbol{u}_1,\boldsymbol{u}_1) & (\boldsymbol{u}_2,\boldsymbol{u}_1) \\ (\boldsymbol{u}_1,\boldsymbol{u}_2) & (\boldsymbol{u}_2,\boldsymbol{u}_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{a},\boldsymbol{u}_1) \\ (\boldsymbol{a},\boldsymbol{u}_2) \end{pmatrix}$$

Solutions This problem is same as Exercise 4.

Problem 4 (10 points) Calculate the Fourier series of the function

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

on the range $[-\pi,\pi]$ in the following form.

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

Solution

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\mathrm{d}x \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx \mathrm{d}x \\ &= \frac{2\{1-(-1)^k\}}{k\pi} \end{aligned}$$

So

$$\sum_{k=1}^{n} b_k \sin kx$$

is the closest linear combination of the functions

$$\frac{1}{2}, \sin x, \cos x, \ldots, \sin nx, \cos nx$$

to the given function f. The Fourier series of f is the limit of the above linear combination as n goes to positive infinity:

$$\sum_{k=1}^{\infty} b_k \sin kx = \sum_{k=1}^{\infty} \frac{2\{1-(-1)^k\}}{k\pi} \sin kx.$$