

# Example 8

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**Problem** Calculate  $T_3(x)$ .

**Solution 1**

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) \\ &= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\ &= (\cos^2 \theta - \sin^2 \theta) \cos \theta - 2 \sin \theta \cos \theta \sin \theta \\ &= (2 \cos^2 \theta - 1) \cos \theta - 2 \sin^2 \theta \cos \theta \\ &= (2 \cos^2 \theta - 1) \cos \theta - 2(1 - \cos^2 \theta) \cos \theta \\ &= 2 \cos^3 \theta - \cos \theta - 2 \cos \theta + 2 \cos^3 \theta \\ &= 4 \cos^3 \theta - 3 \cos \theta\end{aligned}$$

Thus we obtain  $T_3(x) = 4x^3 - 3x$ .

**Solution 2** By applying the recurrence formula of Chebyshev polynomials

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$$

to  $T_2(x) = 2x^2 - 1$  and  $T_1(x) = x$ , we obtain  $T_3(x)$  as follows.

$$\begin{aligned}T_3(x) &= 2xT_2(x) - T_1(x) \\ &= 2x(2x^2 - 1) - x \\ &= 4x^3 - 2x - x \\ &= 4x^3 - 3x\end{aligned}$$