Example 13-2

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Example

(1) Calculate the Fourier series $f(x) = x$ on the range $[-L, L]$ in the form of
$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}\{a_k \cos \omega_k x + b_k \sin \omega_k x\}$$
where $\omega_k = \dfrac{k\pi}{L}$.

(2) Calculate the Fourier series $f(x) = x$ on the range $[-L, L]$ in the form of
$$\sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}$$
where $\omega_k = \dfrac{k\pi}{L}$.

Solution

(1) Assume the following equation holds. (Note that there are no coefficients a_0, …, a_n that satisfy the equation, but it's ok.)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\{a_k \cos \omega_k x + b_k \sin \omega_k x\}$$

Firstly integrate the both sides on $[-L, L]$.

$$\begin{aligned}\int_{-L}^{L} f(x)\mathrm{d}x &= \int_{-L}^{L} \frac{1}{2}a_0 \mathrm{d}x \\ &= La_0\end{aligned}$$

So we obtain a_0 as follows.

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x \\
&= \frac{1}{L}\int_{-L}^{L} x\mathrm{d}x \\
&= 0
\end{aligned}
$$

Secondly we multiply the both sides by $\cos\omega_k x$ and integrate them on the range $[-L, L]$.

$$
\begin{aligned}
\int_{-L}^{L} f(x)\cos\omega_k x\mathrm{d}x &= \int_{-L}^{L} a_k \cos^2\omega_k x\mathrm{d}x \\
&= La_k
\end{aligned}
$$

So we calculate a_k as follows.

$$
\begin{aligned}
a_k &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\omega_k x\mathrm{d}x \\
&= \frac{1}{L}\int_{-L}^{L} x\cos\omega_k x\mathrm{d}x \\
&= 0
\end{aligned}
$$

Thirdly we multiply the both sides by $\sin\omega_k x$ and integrate them on the range $[-L, L]$.

$$
\begin{aligned}
\int_{-L}^{L} f(x)\sin\omega_k x\mathrm{d}x &= \int_{-L}^{L} b_k \sin^2\omega_k x\mathrm{d}x \\
&= Lb_k
\end{aligned}
$$

So we calculate b_k as follows.

$$\begin{aligned}
b_k &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\omega_k x \mathrm{d}x \\
&= \frac{1}{L}\int_{-L}^{L} x\sin\omega_k x \mathrm{d}x \\
&= \frac{1}{L}\left\{\left[x\frac{-\cos\omega_k x}{\omega_k}\right]_{-L}^{L} - \int_{-L}^{L}\frac{-\cos\omega_k x}{\omega_k}\mathrm{d}x\right\} \\
&= \frac{1}{L}\left\{L\cdot\frac{-\cos k\pi}{\omega_k} - (-L)\cdot\frac{-\cos(-k\pi)}{\omega_k}\right\} \\
&= \frac{1}{L}\left\{L\cdot\frac{-\cos k\pi}{\omega_k} + L\cdot\frac{-\cos k\pi}{\omega_k}\right\} \\
&= -\frac{2\cos k\pi}{\omega_k} \\
&= -\frac{2}{\omega_k}(-1)^k
\end{aligned}$$

So the following is the linear combination closest to the function $f(x) = x$.

$$\sum_{k=1}^{n}\{-\frac{2}{\omega_k}(-1)^k\sin\omega_k x\}$$

The Fourier series is the limit of the linear combination as n goes to infinity.

$$\sum_{k=1}^{\infty}\{-\frac{2}{\omega_k}(-1)^k\sin\omega_k x\}$$

We depict the partial summation of this series up to the term of $\cos\omega_{10}x$ and $f(x) = x$ when $L = 1.0$ in Fig. 1.

(2) Assume the following equation holds. (Note that there are no coefficients c_{-n}, $\ldots$, c_0, $\ldots$, c_n that satisfy the equation, but it's ok.)

$$f(x) = \sum_{l=-n}^{n} c_l e^{i\omega_l x}$$

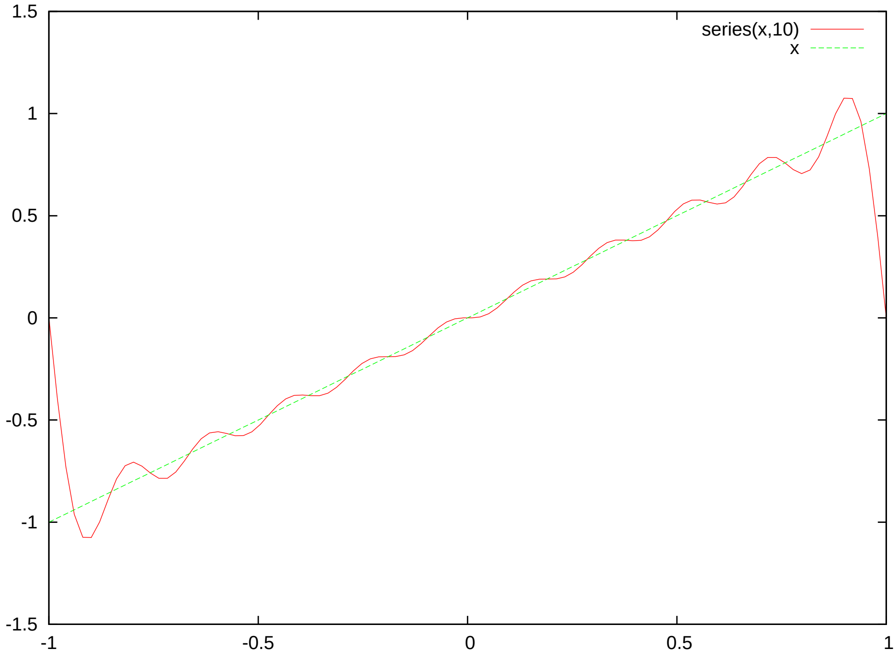

Figure 1: Comparison between the function $f(x) = x$ and the partial sum up to the term of $\cos \omega_{10} x$

Multiply the both sides of this equation by $e^{-i\omega_k x}$ and integrate them on the range $[-L, L]$.

$$\begin{aligned}
\int_{-L}^{L} f(x)e^{-i\omega_k x}\mathrm{d}x &= \int_{-L}^{L} e^{-i\omega_k x}\sum_{l=-n}^{n} c_l e^{i\omega_l x}\mathrm{d}x \\
&= \int_{-L}^{L} \sum_{l=-n}^{n} c_l e^{i\omega_l x} e^{-i\omega_k x}\mathrm{d}x \\
&= \int_{-L}^{L} \sum_{l=-n}^{n} c_l e^{i(\omega_l-\omega_k)x}\mathrm{d}x \\
&= \int_{-L}^{L} \sum_{l=-n}^{n} c_l e^{i\omega_{l-k}x}\mathrm{d}x \\
&\qquad\qquad (\text{since } \omega_l - \omega_k = \frac{l\pi}{L} - \frac{k\pi}{L} = \frac{(l-k)\pi}{L}) \\
&= \sum_{l=-n}^{n} \int_{-L}^{L} c_l e^{i\omega_{l-k}x}\mathrm{d}x \\
&= \int_{-L}^{L} c_k e^0 \mathrm{d}x \\
&= \int_{-L}^{L} c_k \mathrm{d}x \\
&= 2Lc_k
\end{aligned}$$

So we obtain c_k as follows.

$$c_k = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\omega_k x}\mathrm{d}x$$

When $k \neq 0$ we calculate c_k as follows.

$$\begin{aligned}
c_k &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\omega_k x}\mathrm{d}x \\
&= \frac{1}{2L}\int_{-L}^{L} xe^{-i\omega_k x}\mathrm{d}x \\
&= \frac{1}{2L}\left\{\left[x\frac{e^{-i\omega_k x}}{-i\omega_k}\right]_{-L}^{L} - \int_{-L}^{L}\frac{e^{-i\omega_k x}}{-i\omega_k}\mathrm{d}x\right\} \\
&= \frac{1}{2L}\cdot\frac{Le^{-i\omega_k L} - (-L)e^{i\omega_k L}}{-i\omega_k} \qquad \left(\text{since } \int_{-L}^{L} e^{-i\omega_k x}\mathrm{d}x = 0\right) \\
&= \frac{1}{2L}\cdot\frac{Le^{-i\frac{k\pi}{L}L} - (-L)e^{i\frac{k\pi}{L}L}}{-i\omega_k} \\
&= \frac{1}{2L}\cdot\frac{Le^{-ik\pi} - (-L)e^{ik\pi}}{-i\omega_k} \\
&= \frac{1}{2L}\cdot\frac{L(-1)^k + L(-1)^k}{-i\omega_k} \\
&= \frac{1}{2L}\cdot\frac{2L(-1)^k}{-i\omega_k} \\
&= \frac{(-1)^k}{-i\omega_k} \\
&= \frac{1}{\omega_k}(-1)^k i
\end{aligned}$$

When $k = 0$ we calculate c_0 as follows.

$$\begin{aligned}
c_0 &= \frac{1}{2L}\int_{-L}^{L} x\mathrm{d}x \\
&= 0
\end{aligned}$$

So we obtain the series

$$\sum_{k=-n}^{n} c_k e^{ikx}$$

where

$$c_k = \begin{cases} \dfrac{1}{\omega_k}(-1)^k i & k \neq 0 \\ 0 & k = 0. \end{cases}$$

The Fourier series is the limit of the above linear combination as n goes to infinity.

$$\lim_{n\to\infty} \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$