Classification of quadratic functions of two variables

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Caution This material concerns topics that are out of the scope of this class. This is for only those who are interested in classification of quadratic functions of two variables. It is required for you to have learned eigenvalues, eigenvectors, and diagonalizing matrices by orthogonal matrices.

The function J for measuring the distance between the line and the points in Exercise 1

$$J = \frac{1}{2}\sum_{i=1}^{3}(ax_i + b - y_i)^2$$

is the following quadratic function of two variables a and b (cf. Solution 2 in Exercise 1).

$$J = \frac{1}{2}\{5a^2 + 3b^2 + 6ab + 8a + 2b + 5\}$$

As I mentioned in the lecture, quadratic functions of two variables are classified into elliptic, parabolic, and hyperbolic types. In this material, we show the function J is elliptic. We let the part other than the $\frac{1}{2}$ in J be $f(a, b)$.

$$f(a, b) = 5a^2 + 3b^2 + 6ab + 8a + 2b + 5$$

The subexpression $5a^2 + 3b^2 + 6ab$ (which is called to have a *quadratic form*) can be represented by using matrices and vectors as follows.

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The subexpression $8a + 2b$ can be represented by using row and column vectors as follows.

$$\begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus the function $f(a, b)$ can be represented as follows.

$$f(a, b) = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + 5$$

We let the matrix $\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$ be A. We have selected A to be a symmetric matrix. Quadratic forms of a and b (terms of a^2, b^2, and ab) can always be represented by using symmetric matrices as in the above. The above expression can be rewritten by using A as follows.

$$f(a,b) = \begin{pmatrix} a & b \end{pmatrix} A \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + 5$$

We then diagonalize the matrix A by using orthogonal matrices. For that purpose we calculate x, y, and λ that satisfies the equation

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

and the inequation

$$\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The equation $A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ can be represented by using the identity matrix I as follows.

$$(\lambda I - A) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since

$$|\lambda I - A| = 0$$

the following equation holds.

$$(\lambda - 5)(\lambda - 3) - 9 = 0$$

By solving this we obtain

$$\lambda = 4 \pm \sqrt{10}.$$

When $\lambda = 4 + \sqrt{10}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 + \sqrt{10} \end{pmatrix}$$

is obtained as a solution and by normalizing this vector we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{20 - 2\sqrt{10}}} \begin{pmatrix} 3 \\ -1 + \sqrt{10} \end{pmatrix}.$$

When $\lambda = 4 - \sqrt{10}$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 - \sqrt{10} \end{pmatrix}$$

is obtained as a solution and by normalizing this vector we obtain

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{20 + 2\sqrt{10}}} \begin{pmatrix} 3 \\ -1 - \sqrt{10} \end{pmatrix}.$$

We make a matrix consisting of the above two unit vectors as follows.

$$U = \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} & \frac{3}{\sqrt{20+2\sqrt{10}}} \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} & \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} \end{pmatrix}$$

Note that this matrix U is a orthogonal matrix. By using the matrix U, the following equation holds.

$$A = U \begin{pmatrix} 4+\sqrt{10} & 0 \\ 0 & 4-\sqrt{10} \end{pmatrix} U^{\mathrm{T}}$$

Here we write the transposed matrix of U as U^{T}. By letting $\lambda_1 = 4+\sqrt{10}$ and $\lambda_2 = 4-\sqrt{10}$ the above equation can be rewritten as follows.

$$A = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{\mathrm{T}}$$

Note that the following equation also holds.

$$U^{\mathrm{T}} A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

By letting

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = U^{\mathrm{T}} \begin{pmatrix} a \\ b \end{pmatrix},$$

since U is a orthogonal matrix, the following equation holds.

$$U \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Thus the quadratic form $5a^2 + 3b^2 + 6ab$ can be rewritten as follows.

$$\begin{aligned} 5a^2 + 3b^2 + 6ab &= \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} a & b \end{pmatrix} A \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \left(\begin{pmatrix} a \\ b \end{pmatrix}, A \begin{pmatrix} a \\ b \end{pmatrix} \right) \end{aligned}$$

(obtained by representing a product of row and column vectors by an inner product)

$$\begin{aligned} &= \left(U \begin{pmatrix} a' \\ b' \end{pmatrix}, AU \begin{pmatrix} a' \\ b' \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} a' \\ b' \end{pmatrix}, U^{\mathrm{T}} AU \begin{pmatrix} a' \\ b' \end{pmatrix} \right) \end{aligned}$$

(obtained by the equality $(Ax, y) = (x, A^{\mathrm{T}} y)$)

$$
\begin{aligned}
&= \left(\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} a' \\ b' \end{pmatrix}, \begin{pmatrix} \lambda_1 a' \\ \lambda_2 b' \end{pmatrix} \right) \\
&= \lambda_1 a'^2 + \lambda_2 b'^2
\end{aligned}
$$

We call the obtained expression $\lambda_1 a'^2 + \lambda_2 b'^2$ as the normal form of the quadratic form $5a^2 + 3b^2 + 6ab$.

The subexpression $8a + 2b$ can be rewritten as follows.

$$
\begin{aligned}
8a + 2b &= \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\
&= \begin{pmatrix} 8 & 2 \end{pmatrix} U \begin{pmatrix} a' \\ b' \end{pmatrix} \\
&= \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} & \frac{3}{\sqrt{20+2\sqrt{10}}} \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} & \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} \\
&= \begin{pmatrix} 8 & 2 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} a' + \frac{3}{\sqrt{20+2\sqrt{10}}} b' \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b' \end{pmatrix} \\
&= \frac{24}{\sqrt{20-2\sqrt{10}}} a' + \frac{24}{\sqrt{20+2\sqrt{10}}} b' + \frac{-2+2\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{-2-2\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b' \\
&= \frac{22+2\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{22-2\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b'
\end{aligned}
$$

So the function $f(a, b)$ can be rewritten as follows.

$$
f(a, b) = \lambda_1 a'^2 + \lambda_2 b'^2 + \frac{22+2\sqrt{10}}{\sqrt{20-2\sqrt{10}}} a' + \frac{22-2\sqrt{10}}{\sqrt{20+2\sqrt{10}}} b' + 5
$$

By completing the squares with respect to a' and b' we obtain

$$
f(a, b) = \lambda_1 \left(a' + \frac{11+\sqrt{10}}{\lambda_1 \sqrt{20-2\sqrt{10}}} \right)^2 + \lambda_2 \left(b' + \frac{11-\sqrt{10}}{\lambda_2 \sqrt{20+2\sqrt{10}}} \right)^2 + \frac{1}{6}.
$$

Since $J = \frac{1}{2} f(a, b)$ we obtain

$$
J = \frac{\lambda_1}{2} \left(a' + \frac{11+\sqrt{10}}{\lambda_1 \sqrt{20-2\sqrt{10}}} \right)^2 + \frac{\lambda_2}{2} \left(b' + \frac{11-\sqrt{10}}{\lambda_2 \sqrt{20+2\sqrt{10}}} \right)^2 + \frac{1}{12}.
$$

Since $\lambda_1 > 0$ and $\lambda_2 > 0$, $\frac{\lambda_1}{2} > 0$ and $\frac{\lambda_2}{2} > 0$ hold and thus the function J is elliptic. So when

$$
\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} -\frac{11+\sqrt{10}}{\lambda_1 \sqrt{20-2\sqrt{10}}} \\ -\frac{11-\sqrt{10}}{\lambda_2 \sqrt{20+2\sqrt{10}}} \end{pmatrix}
$$

J takes a minimum value of $\frac{1}{12}$. Since $\begin{pmatrix} a \\ b \end{pmatrix} = U \begin{pmatrix} a' \\ b' \end{pmatrix}$ we obtain

$$\begin{aligned}
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} \frac{3}{\sqrt{20-2\sqrt{10}}} & \frac{3}{\sqrt{20+2\sqrt{10}}} \\ \frac{-1+\sqrt{10}}{\sqrt{20-2\sqrt{10}}} & \frac{-1-\sqrt{10}}{\sqrt{20+2\sqrt{10}}} \end{pmatrix} \begin{pmatrix} -\frac{11+\sqrt{10}}{\lambda_1\sqrt{20-2\sqrt{10}}} \\ -\frac{11-\sqrt{10}}{\lambda_2\sqrt{20+2\sqrt{10}}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{2} \\ \frac{7}{6} \end{pmatrix}.
\end{aligned}$$

So when $a = -\frac{3}{2}$ and $b = \frac{7}{6}$ the function J takes a minimum value of $\frac{1}{12}$.

Note The function J is obtained by linear transforming the elliptic quadratic function of two variables

$$\frac{\lambda_1}{2}a^2 + \frac{\lambda_2}{2}b^2$$

by using the matrix U^{T}, which represents a rotation, and parallel shifting $-\frac{3}{2}$ in the a-axis, $\frac{7}{6}$ in the b-axis, and $\frac{1}{12}$ in the J-axis.