

An introduction to Fourier transforms

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This document is largely based on the reference book [1] with some parts slightly changed.

1 Fourier Integral

Suppose a periodic function $f_L(x)$ of period $2L$ is represented by a Fourier series

$$f_L(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos \omega_k x + b_k \sin \omega_k x)$$

where a_0, a_k, b_k, ω_k are given as follows.

$$\begin{aligned}\omega_k &= \frac{k\pi}{L} \\ a_0 &= \frac{1}{L} \int_{-L}^L f_L(x) dx \\ a_k &= \frac{1}{L} \int_{-L}^L f_L(x) \cos \omega_k x dx \\ b_k &= \frac{1}{L} \int_{-L}^L f_L(x) \sin \omega_k x dx\end{aligned}$$

Then $f_L(x)$ is represented as follows.

$$f_L(x) = \frac{1}{2L} \int_{-L}^L f_L(x) dx + \frac{1}{L} \sum_{k=1}^{\infty} \left\{ \cos \omega_k x \int_{-L}^L f_L(x) \cos \omega_k x dx + \sin \omega_k x \int_{-L}^L f_L(x) \sin \omega_k x dx \right\}$$

We now set

$$\Delta\omega = \omega_{k+1} - \omega_k = \frac{(k+1)\pi}{L} - \frac{k\pi}{L} = \frac{\pi}{L}.$$

Since $1/L = \Delta\omega/\pi$ we may rewrite the above equation as follows.

$$\begin{aligned} f_L(x) &= \frac{1}{2L} \int_{-L}^L f_L(x) dx + \frac{\Delta\omega}{\pi} \sum_{k=1}^{\infty} \left\{ \cos \omega_k x \int_{-L}^L f_L(x) \cos \omega_k x dx \right. \\ &\quad \left. + \sin \omega_k x \int_{-L}^L f_L(x) \sin \omega_k x dx \right\} \\ &= \frac{1}{2L} \int_{-L}^L f_L(x) dx + \frac{1}{\pi} \sum_{k=1}^{\infty} \left\{ \cos \omega_k x \int_{-L}^L f_L(x) \cos \omega_k x dx \right. \\ &\quad \left. + \sin \omega_k x \int_{-L}^L f_L(x) \sin \omega_k x dx \right\} \Delta\omega \end{aligned}$$

We now let $L \rightarrow \infty$ and assume that the resulting function

$$f(x) = \lim_{L \rightarrow \infty} f_L(x)$$

is *absolutely integrable* on the x -axis; that is, the following finite limits exist.

$$\lim_{a \rightarrow -\infty} \int_a^0 |f(x)| dx + \lim_{a \rightarrow \infty} \int_0^a |f(x)| dx$$

Note that this is written as $\int_{-\infty}^{\infty} |f(x)| dx$. Then the first term $\frac{1}{2L} \int_{-L}^L f_L(x) dx$ approaches zero. Also $\Delta\omega = \pi/L$ approaches zero and it seems *plausible* that the infinite series becomes an integral from 0 to ∞ as follows^{1 2}.

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left\{ \cos \omega x \int_{-\infty}^{\infty} f(x) \cos \omega x dx + \sin \omega x \int_{-\infty}^{\infty} f(x) \sin \omega x dx \right\} d\omega$$

¹It just seems plausible and is not the logical consequence (at least in the explanation in this material). Fourier series and Fourier transformations should be argued separately. Having said that, there are intuitively some connection so that, in textbooks used in Engineering classes, Fourier transformation is explained as the limit of Fourier series as the period goes to infinity.

²Notice that I use the same name x for a few variables each of which has different scope. Of course we can rename inner x as other name such as v , which might be a usual way of writing this kind of formula.

By introducing the functions

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x dx \quad (1)$$

and

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x dx \quad (2)$$

we can also write the above formula as follows.

$$f(x) = \int_0^{\infty} \{A(\omega) \cos \omega x + B(\omega) \sin \omega x\} d\omega \quad (3)$$

This is called a representation of $f(x)$ by a **Fourier integral**.

The following theorem holds (see p. 513 of the reference book [1]).

Theorem 1 *If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if $f(x)$ is absolutely integrable, then $f(x)$ can be represented by (3) with A and B given by (1) and (2). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point. In formula,*

$$\int_0^{\infty} \{A(\omega) \cos \omega x + B(\omega) \sin \omega x\} d\omega = \frac{f(x-0) + f(x+0)}{2}.$$

□

Example Calculate the Fourier integral of the following function.

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution We calculate $A(\omega)$ and $B(\omega)$ as follows.

$$\begin{aligned}
 A(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x dx \\
 &= \frac{1}{\pi} \int_{-1}^1 \cos \omega x dx \\
 &= \frac{1}{\pi} \left[\frac{\sin \omega x}{\omega} \right]_{-1}^1 \\
 &= \frac{2 \sin \omega}{\pi \omega}
 \end{aligned}$$

$$\begin{aligned}
 B(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x dx \\
 &= \frac{1}{\pi} \int_{-1}^1 \sin \omega x dx \\
 &= 0
 \end{aligned}$$

So we obtain the Fourier integral of $f(x)$ as follows.

$$\frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega x \sin \omega}{\omega} d\omega$$

By Theorem 1 we obtain the following equality.

$$\frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega x \sin \omega}{\omega} d\omega = \begin{cases} 1 & -1 < x < 1 \\ 1/2 & x = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

We should mention one thing. The integral above can be considered as the limit of the function

$$\frac{2}{\pi} \int_0^a \frac{\cos \omega x \sin \omega}{\omega} d\omega$$

as a goes to infinity. In this integral, there are oscillations near the points $x = -1$ and $x = 1$. The oscillations does not disappear even if a increases, similarly to the Fourier series. This is called the Gibbs phenomenon.

Note By setting $x = 0$ in the Fourier integral of $f(x)$, we obtain the following equality.

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega}{\omega} d\omega = 1$$

By multiplying the both sides by $\frac{\pi}{2}$ we obtain the following equality.

$$\int_0^{\infty} \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}$$

This is called the **Dirichlet integral**. Consider the following (partial) integral (∞ is replaced by a) so called **sine integral**.

$$\text{Si}(a) = \int_0^a \frac{\sin \omega}{\omega} d\omega$$

In the sine integral $\text{Si}(a)$, there are oscillations. The oscillations in the above integral come from the oscillations in the sine integral.

2 Fourier transform

The (real) Fourier integral is

$$f(x) = \int_0^{\infty} \{A(\omega) \cos \omega x + B(\omega) \sin \omega x\} d\omega$$

where A and B are given as follows.

$$\begin{aligned} A(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv \\ B(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin \omega v dv \end{aligned}$$

Substituting A and B into the integral, we have

$$\begin{aligned}
f(x) &= \int_0^\infty \left\{ \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \omega v dv \cos \omega x + \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin \omega v dv \sin \omega x \right\} d\omega \\
&= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty f(v) \cos \omega v dv \cos \omega x + \int_{-\infty}^\infty f(v) \sin \omega v dv \sin \omega x \right\} d\omega \\
&= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty f(v) \cos \omega v \cos \omega x dv + \int_{-\infty}^\infty f(v) \sin \omega v \sin \omega x dv \right\} d\omega \\
&= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty f(v) (\cos \omega v \cos \omega x + \sin \omega v \sin \omega x) dv \right\} d\omega \\
&= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty f(v) \cos(\omega x - \omega v) dv \right\} d\omega \\
&= \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty f(v) \cos(\omega(x - v)) dv \right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \left\{ \int_{-\infty}^\infty f(v) \cos(\omega(x - v)) dv \right\} d\omega \\
&\quad (\text{since the integral in the brackets is an even function of } \omega)
\end{aligned}$$

The integral of this form with \sin instead of \cos

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left\{ \int_{-\infty}^\infty f(v) \sin(\omega(x - v)) dv \right\} d\omega$$

is zero since the integral in the brackets is an odd function of ω .

By the Euler formula

$$e^{ix} = \cos x + i \sin x$$

we obtain the following equality.

$$e^{i\omega(x-v)} = \cos(\omega(x - v)) + i \sin(\omega(x - v))$$

By multiplying the both sides by $f(v)$ we obtain the following equality.

$$f(v)e^{i\omega(x-v)} = f(v) \cos(\omega(x - v)) + i f(v) \sin(\omega(x - v))$$

By taking integral with respect to v and ω and multiplying the result by $\frac{1}{2\pi}$

we obtain the following equality.

$$\begin{aligned}
& \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) e^{i\omega(x-v)} dv \right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) \cos(\omega(x-v)) + i f(v) \sin(\omega(x-v)) dv \right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) \cos(\omega(x-v)) dv \right\} d\omega \\
&\quad + i \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) \sin(\omega(x-v)) dv \right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) \cos(\omega(x-v)) dv \right\} d\omega \\
&= f(x)
\end{aligned}$$

So we obtain the following equality.

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) e^{i\omega(x-v)} dv \right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) e^{i\omega x - i\omega v} dv \right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) e^{i\omega x} e^{-i\omega v} dv \right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right\} e^{i\omega x} d\omega
\end{aligned}$$

We usually write this as follows³.

$$\begin{aligned}
f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega \\
F(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx
\end{aligned}$$

We call $\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$ the Fourier transform of $f(x)$ and we call $\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$ the inverse Fourier transform of $F(\omega)$. Note that the

³The constants ($\frac{1}{2\pi}$ and 1) depend on textbooks. They may be $\frac{1}{\sqrt{2\pi}}$ and $\frac{1}{\sqrt{2\pi}}$.

above two equalities are just definitions of transformations and $f(x)$ might not be equal to the inverse Fourier transform of the Fourier transform of $f(x)$ (see Theorem 1).

Example Calculate the Fourier transform of the following function.

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$\begin{aligned} F(\omega) &= \int_{-1}^1 e^{-i\omega x} dx \\ &= \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^1 \\ &= \frac{1}{-i\omega} (e^{-i\omega} - e^{i\omega}) \\ &= \frac{1}{-i\omega} (\cos \omega - i \sin \omega - (\cos \omega + i \sin \omega)) \\ &= \frac{1}{-i\omega} (-2i \sin \omega) \\ &= 2 \frac{\sin \omega}{\omega} \end{aligned}$$

References

- [1] Erwin Kreyszig. *Advanced Engineering Mathematics*. John Wiley & Sons Ltd., tenth edition, 2011.