

Example 11

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Example Calculate the value of the following series by using the Parseval's equality for the Fourier series of $f(x) = x$ on the range $[-\pi, \pi]$ following the steps (1)-(5).

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

- (1) Calculate the linear combination of the following orthogonal functions that is closest to the function $f(x)$. As for the measure of the distance, use (the half of) the integral of the square of the difference on the range $[-\pi, \pi]$.

$$\left\{ \frac{1}{2}, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx \right\}$$

- (2) Obtain the Fourier series of $f(x)$ on the range $[-\pi, \pi]$. (The Fourier series of $f(x) = x$ is the limit of the linear combination obtained in (1) as n goes to infinity.)
- (3) Normalise the series obtained in (2).
- (4) Write down the Parseval's equality for the series obtained in (4).
- (5) Calculate the value of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Note that in an inner product space \mathcal{L} , when the approximation, in the sense of the least square, of $\mathbf{u} \in \mathcal{L}$ by a linear combination of an orthonormal basis $\{\mathbf{e}_i | i \geq 1\}$ in \mathcal{L}

$$\sum_{k=1}^n c_k \mathbf{e}_k$$

converges to \mathbf{u} in the sense that the norm of the difference converges to 0 as n goes to infinity, the following equation, called Parseval's equality, holds.

$$\|\mathbf{u}\|^2 = \sum_{k=1}^{\infty} c_k^2$$

Solution

(1) Assume the following equation holds. (Note: There are no coefficients $a_0, \dots, a_n, b_1, \dots, b_n$ that satisfy the equation, but it's ok.)

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \quad (1)$$

Integrate the both sides of the equation (1) on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} \frac{1}{2} a_0 dx \\ &= a_0 \pi \end{aligned}$$

Then we calculate a_0 as follows.

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x dx \\ &= 0 \end{aligned}$$

Multiply the both sides of the equation (1) by $\cos kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos kx dx &= a_k \int_{-\pi}^{\pi} \cos^2 kx dx \\ &= a_k \pi \end{aligned}$$

Then we calculate a_k as follows.

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos kx dx \\ &= 0 \quad (\text{since } x \cos kx \text{ is an odd function}) \end{aligned}$$

Multiply the both sides of the equation (1) by $\sin kx$ and integrate them on the range $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \sin kx dx &= b_k \int_{-\pi}^{\pi} \sin^2 kx dx \\ &= b_k \pi\end{aligned}$$

Then we calculate b_k as follows.

$$\begin{aligned}b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx dx\end{aligned}$$

Here we calculate the integral $\int_{-\pi}^{\pi} x \sin kx dx$.

$$\begin{aligned}\int_{-\pi}^{\pi} x \sin kx dx &= \left[x \frac{-\cos kx}{k} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-\cos kx}{k} dx \\ &= -\frac{1}{k} [x \cos kx]_{-\pi}^{\pi} \quad (\text{since } \int_{-\pi}^{\pi} \frac{-\cos kx}{k} dx \text{ is } 0) \\ &= -\frac{1}{k} (\pi \cos \pi k - (-\pi) \cos(-\pi k)) \\ &= -\frac{1}{k} (\pi \cos \pi k + \pi \cos \pi k) \\ &= -\frac{2\pi}{k} \cos \pi k \\ &= -\frac{2\pi}{k} (-1)^k\end{aligned}$$

We resume the calculation of b_k .

$$\begin{aligned}b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin kx dx \\ &= \frac{1}{\pi} \cdot -\frac{2\pi}{k} (-1)^k \\ &= -\frac{2}{k} (-1)^k\end{aligned}$$

So the linear combination that is closest to the function $f(x)$ is

$$\sum_{k=1}^n -\frac{2}{k} (-1)^k \sin kx.$$

(2) The Fourier expansion of $f(x) = x$ is the limit of the above linear combination as n goes to infinity:

$$\sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx.$$

(3) Firstly we calculate the norm of $\sin kx$.

$$\begin{aligned} \|\sin kx\| &= \sqrt{(\sin kx, \sin kx)} \\ &= \sqrt{\int_{-\pi}^{\pi} \sin^2 kx dx} \\ &= \sqrt{\pi} \end{aligned}$$

So the Fourier series of $f(x) = x$ is normalized as follows.

$$\sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sin kx = \sum_{k=1}^{\infty} -\frac{2}{k}(-1)^k \sqrt{\pi} \cdot \frac{\sin kx}{\sqrt{\pi}}$$

(4) By the Parseval's equality, we obtain the following equation.

$$\|f\|^2 = \sum_{k=1}^{\infty} \left(-\frac{2}{k}(-1)^k \sqrt{\pi} \right)^2 \quad (2)$$

The left hand side of the equation (2) is calculated as follows.

$$\begin{aligned} \|f\|^2 &= (f, f) \\ &= \int_{-\pi}^{\pi} f(x)^2 dx \\ &= \int_{-\pi}^{\pi} x^2 dx \\ &= \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} \\ &= \left(\frac{\pi^3}{3} - \left(-\frac{\pi^3}{3} \right) \right) \\ &= \frac{2}{3}\pi^3 \end{aligned}$$

The right hand side of the equation (2) is calculated as follows.

$$\begin{aligned}
\text{RHS} &= \sum_{k=1}^{\infty} \left(-\frac{2}{k} (-1)^k \sqrt{\pi} \right)^2 \\
&= \sum_{k=1}^{\infty} \frac{4\pi}{k^2} ((-1)^k)^2 \\
&= \sum_{k=1}^{\infty} \frac{4\pi}{k^2} (-1)^{2k} \\
&= \sum_{k=1}^{\infty} \frac{4\pi}{k^2} ((-1)^2)^k \\
&= \sum_{k=1}^{\infty} \frac{4\pi}{k^2} 1^k \\
&= \sum_{k=1}^{\infty} \frac{4\pi}{k^2} \\
&= 4\pi \sum_{k=1}^{\infty} \frac{1}{k^2}
\end{aligned}$$

So the Parseval's equality for the Fourier series of $f(x) = x$ is obtained as follows.

$$\frac{2}{3}\pi^3 = 4\pi \sum_{k=1}^{\infty} \frac{1}{k^2}$$

(5) So the value of the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is obtained as follows.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4\pi} \cdot \frac{2}{3}\pi^3 = \frac{1}{6}\pi^2$$

(Note) The obtained value $\frac{1}{6}\pi^2$ is the value of the zeta function $\zeta(n)$ when $n = 2$. The zeta function is given as follows.

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

$$\text{So } \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{6}\pi^2.$$