

Exercise 3

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Exercise 3 Fit a parabola (a square function) to the function $\cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so that (the half of) the integral of the squares of the distances between them, where the distance is measured in the vertical direction (the y-direction).

Solution Let the function be $f(x) = ax^2 + bx + c$. The half of the integral of the squares of the distances between $f(x)$ and $\cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is given as follows.

$$\begin{aligned} J &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{f(x) - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \end{aligned}$$

J takes the minimum value in the point where the partial derivatives of J with respect to a , b , and c are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial c} = 0$$

Firstly the partial derivative of J with respect to a is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial a} &= \frac{\partial}{\partial a} \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \frac{\partial}{\partial a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial a} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\}x^2 dx \end{aligned}$$

$$\begin{aligned}
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}x^2 dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^4 + bx^3 + cx^2 - x^2 \cos x\} dx \\
&= a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4 dx + b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x dx
\end{aligned}$$

Here we calculate each of the integrals. As for x^4 we obtain

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4 dx &= 2 \int_0^{\frac{\pi}{2}} x^4 dx \quad (\text{since } x^4 \text{ is an even function}) \\
&= 2 \left[\frac{x^5}{5} \right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi^5}{80}
\end{aligned}$$

As for x^3 its integral is 0 since it is an odd function.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx = 0$$

As for x^2 we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx = 2 \int_0^{\frac{\pi}{2}} x^2 dx = 2 \left[\frac{x^3}{3} \right]_0^{\frac{\pi}{2}} = 2 \cdot \frac{\pi^3}{24} = \frac{\pi^3}{12}$$

As for $x^2 \cos x$ we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x dx = 2 \int_0^{\frac{\pi}{2}} x^2 \cos x dx \quad (\text{since } x^2 \cos x \text{ is an even function})$$

In the following we calculate $\int_0^{\frac{\pi}{2}} x^2 \cos x dx$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x^2 \cos x dx &= \left[x^2 \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x \sin x dx \\
&= \frac{\pi^2}{4} - 2 \int_0^{\frac{\pi}{2}} x \sin x dx
\end{aligned}$$

Here we calculate $\int_0^{\frac{\pi}{2}} x \sin x dx$.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x \sin x dx &= \left[x \frac{\cos x}{-1} \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\cos x}{-1} dx \\
&= [\sin x]_0^{\frac{\pi}{2}} \\
&= 1
\end{aligned}$$

Now we resume the calculation of $\int_0^{\frac{\pi}{2}} x^2 \cos x dx$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x^2 \cos x dx &= \frac{\pi^2}{4} - 2 \cdot 1 \\ &= \frac{\pi^2}{4} - 2\end{aligned}$$

Thus $\frac{\partial J}{\partial a}$ is obtained as follows.

$$\begin{aligned}\frac{\partial J}{\partial a} &= \frac{\pi^5}{80}a + \frac{\pi^3}{12}c - 2\left(\frac{\pi^2}{4} - 2\right) \\ &= \frac{\pi^5}{80}a + \frac{\pi^3}{12}c - \frac{\pi^2}{2} + 4\end{aligned}$$

Secondly the partial derivative of J with respect to b is calculated as follows.

$$\begin{aligned}\frac{\partial J}{\partial b} &= \frac{\partial}{\partial b} \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \frac{\partial}{\partial b} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial b} \{ax^2 + bx + c - \cos x\}^2 dx \\ &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\}x dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}x dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^3 + bx^2 + cx - x \cos x\} dx \\ &= a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 dx + b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx + c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx \\ &= 2b \int_0^{\frac{\pi}{2}} x^2 dx \quad (x^3, x, \text{ and } x \cos x \text{ are odd functions and } x^2 \text{ is an even function}) \\ &= 2b \cdot \frac{\pi^3}{24} \\ &= \frac{\pi^3}{12}b\end{aligned}$$

Thirdly the partial derivative of J with respect to c is calculated as follows.

$$\frac{\partial J}{\partial c} = \frac{\partial}{\partial c} \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx$$

$$\begin{aligned}
&= \frac{1}{2} \frac{\partial}{\partial c} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 dx \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial c} \{ax^2 + bx + c - \cos x\}^2 dx \\
&= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\} \cdot 1 \, dx \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\} dx \\
&= a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 dx + b \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx + c \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx \\
&= 2a \int_0^{\frac{\pi}{2}} x^2 dx + \pi c - 2 \int_0^{\frac{\pi}{2}} \cos x dx \\
&= 2a \cdot \frac{\pi^3}{24} + \pi c - 2 [\sin x]_0^{\frac{\pi}{2}} \\
&= \frac{\pi^3}{12} a + \pi c - 2
\end{aligned}$$

Thus we obtain the system of equations with respect to a , b , and c .

$$\begin{aligned}
\frac{\pi^5}{80} a + \frac{\pi^3}{12} c - \frac{\pi^2}{2} + 4 &= 0 \\
\frac{\pi^3}{12} b &= 0 \\
\frac{\pi^3}{12} a + \pi c - 2 &= 0
\end{aligned}$$

By solving this, we obtain the solution.

$$a = \frac{60\pi^2 - 720}{\pi^5}, \quad b = 0, \quad c = \frac{60 - 3\pi^2}{\pi^3}$$

Hence the function is obtained as follows.

$$f(x) = \frac{60\pi^2 - 720}{\pi^5} x^2 + \frac{60 - 3\pi^2}{\pi^3}$$

The function is depicted with $\cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ in Fig. 1. In Fig. 1 the red curve is the square function and the green curve is the function $\cos x$.

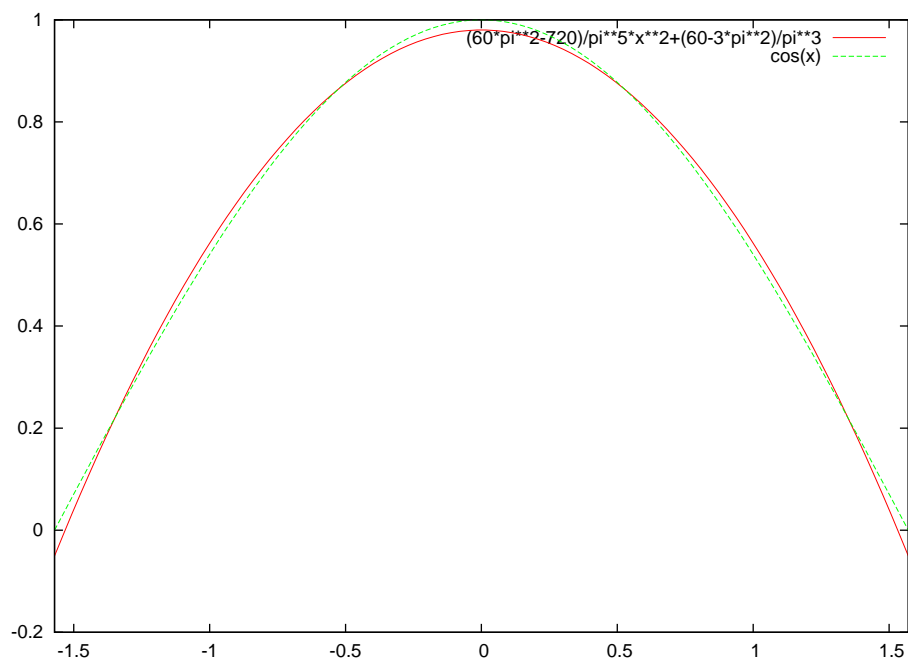


Figure 1: The closest square function to $\cos x$ on the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$