

## Three solutions for Exercise 10-2

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**Exercise** Consider the set of real-valued continuous functions on the interval  $[-\pi, \pi]$ . As we did in Exercise 9-1, we construct an inner product space where the addition, scalar multiplication, and inner product are defined as follows.

$$\begin{aligned}(\mathbf{f} + \mathbf{g})(x) &= f(x) + g(x) \\(c\mathbf{f})(x) &= c(f(x)) \\(\mathbf{f}, \mathbf{g}) &= \int_{-\pi}^{\pi} f(x)g(x)dx\end{aligned}$$

On this inner product space, approximate a function  $f(x) = x^2$  by a linear combination of the functions  $e_1(x) = \frac{1}{2}$ ,  $e_2(x) = \cos x$ , and  $e_3(x) = \sin x$  (i.e.,  $\sum_{k=1}^3 c_k e_k(x) = c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$  for some  $c_1$ ,  $c_2$ , and  $c_3$ ). That is, obtain  $c_1$ ,  $c_2$ ,  $c_3$  so that  $c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$  is closest to  $f(x)$ . As for the measure of the distance, use (the half of) the square of the norm of the difference of  $c_1 e_1(x) + c_2 e_2(x) + c_3 e_3(x)$  and  $f(x)$ .

$$J = \frac{1}{2} \left\| \sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f} \right\|^2$$

The norm is defined as follows.

$$\|\mathbf{f}\| = \sqrt{(\mathbf{f}, \mathbf{f})} = \sqrt{\int_{-\pi}^{\pi} f(x)^2 dx}$$

We show three solutions. One is by solving  $\frac{\partial J}{\partial c_i} = 0$  for  $i = 1, 2, 3$ . Another is by considering the special case. The other is by calculating the projection of  $\mathbf{f}$  on the subspace spanned by  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ .

**Solution 1** Firstly calculate  $J$  as follows.

$$\begin{aligned}
J &= \frac{1}{2} \left\| \sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f} \right\|^2 \\
&= \frac{1}{2} \left( \sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f}, \sum_{k=1}^3 c_k \mathbf{e}_k - \mathbf{f} \right) \\
&= \frac{1}{2} \left\{ \left( \sum_{k=1}^3 c_k \mathbf{e}_k, \sum_{k=1}^3 c_k \mathbf{e}_k \right) - 2 \left( \mathbf{f}, \sum_{k=1}^3 c_k \mathbf{e}_k \right) + \|\mathbf{f}\|^2 \right\} \\
&= \frac{1}{2} \left\{ \sum_{k,l=1}^3 c_k c_l (\mathbf{e}_k, \mathbf{e}_l) - 2 \sum_{k=1}^3 c_k (\mathbf{f}, \mathbf{e}_k) + \|\mathbf{f}\|^2 \right\}
\end{aligned}$$

Partially differentiate this with respect to  $c_i$  ( $i = 1, 2, 3$ ).

$$\begin{aligned}
\frac{\partial J}{\partial c_i} &= \frac{\partial}{\partial c_i} \frac{1}{2} \left\{ \sum_{k,l=1}^3 c_k c_l (\mathbf{e}_k, \mathbf{e}_l) - 2 \sum_{k=1}^3 c_k (\mathbf{f}, \mathbf{e}_k) + \|\mathbf{f}\|^2 \right\} \\
&= \frac{1}{2} \left\{ \frac{\partial}{\partial c_i} \sum_{k,l=1}^3 c_k c_l (\mathbf{e}_k, \mathbf{e}_l) - 2 \frac{\partial}{\partial c_i} \sum_{k=1}^3 c_k (\mathbf{f}, \mathbf{e}_k) \right\} \\
&= \frac{1}{2} \left\{ 2 \sum_{k=1}^3 c_k (\mathbf{e}_k, \mathbf{e}_i) - 2 (\mathbf{f}, \mathbf{e}_i) \right\} \\
&= \sum_{k=1}^3 c_k (\mathbf{e}_k, \mathbf{e}_i) - (\mathbf{f}, \mathbf{e}_i)
\end{aligned}$$

By writing  $\frac{\partial J}{\partial c_i} = 0$  for  $i = 1, 2, 3$  in matrix form we obtain

$$\begin{pmatrix} (\mathbf{e}_1, \mathbf{e}_1) & (\mathbf{e}_2, \mathbf{e}_1) & (\mathbf{e}_3, \mathbf{e}_1) \\ (\mathbf{e}_1, \mathbf{e}_2) & (\mathbf{e}_2, \mathbf{e}_2) & (\mathbf{e}_3, \mathbf{e}_2) \\ (\mathbf{e}_1, \mathbf{e}_3) & (\mathbf{e}_2, \mathbf{e}_3) & (\mathbf{e}_3, \mathbf{e}_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Since  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  are orthogonal to each other, we obtain

$$\begin{pmatrix} \|\mathbf{e}_1\|^2 & 0 & 0 \\ 0 & \|\mathbf{e}_2\|^2 & 0 \\ 0 & 0 & \|\mathbf{e}_3\|^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Hence we obtain  $c_i = \frac{(\mathbf{f}, \mathbf{e}_i)}{\|\mathbf{e}_i\|^2}$  for  $i = 1, 2, 3$ .

$$\begin{aligned}
(\mathbf{f}, \mathbf{e}_1) &= \int_{-\pi}^{\pi} \frac{x^2}{2} dx = \left[ \frac{x^3}{6} \right]_{-\pi}^{\pi} = \frac{\pi^3}{3} \\
(\mathbf{f}, \mathbf{e}_2) &= \int_{-\pi}^{\pi} x^2 \cos x dx \\
&= \left[ x^2 \frac{\sin x}{-1} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin x}{-1} dx \\
&= 2 \int_{-\pi}^{\pi} x \sin x dx \\
&= 2 \left\{ [x \cos x]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos x dx \right\} \\
&= 2 \left\{ \pi \cos \pi - (-\pi) \cos(-\pi) \right\} \\
&= 2 \{ 2\pi \cos \pi \} \\
&= 4\pi \cos \pi \\
&= -4\pi \\
(\mathbf{f}, \mathbf{e}_3) &= \int_{-\pi}^{\pi} x^2 \sin x dx = 0 \\
(\mathbf{e}_1, \mathbf{e}_1) &= \int_{-\pi}^{\pi} \frac{1}{4} dx = \left[ \frac{x}{4} \right]_{-\pi}^{\pi} = \frac{\pi}{2} \\
(\mathbf{e}_2, \mathbf{e}_2) &= \int_{-\pi}^{\pi} \cos^2 x dx = \pi
\end{aligned}$$

So we obtain

$$\begin{aligned}
c_1 &= \frac{\frac{\pi^3}{3}}{\frac{\pi}{2}} = \frac{2}{3}\pi^2 \\
c_2 &= \frac{-4\pi}{\pi} = -4 \\
c_3 &= 0
\end{aligned}$$

Thus the linear combination is obtained as follows.

$$\frac{2}{3}\pi^2 \mathbf{e}_1 - 4\mathbf{e}_2$$

This vector represents the following function.

$$\left( \frac{2}{3}\pi^2 \mathbf{e}_1 - 4\mathbf{e}_2 \right)(x) = \frac{2}{3}\pi^2 e_1(x) - 4e_2(x)$$

$$= \frac{1}{3}\pi^2 - 4\cos x$$

Notice that the function consists of the terms up to the terms of  $\cos x$  and  $\sin x$  in the Fourier series of  $f(x)$ .

**Solution 2** Assume the following equation holds.

$$f = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$$

(There is actually no coefficients  $c_1$ ,  $c_2$ , and  $c_3$  that satisfy the equation, but it's ok.) Take the inner product with  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  in the both side.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) &= c_1(\mathbf{e}_1, \mathbf{e}_1) + c_2(\mathbf{e}_2, \mathbf{e}_1) + c_3(\mathbf{e}_3, \mathbf{e}_1) \\(\mathbf{f}, \mathbf{e}_2) &= c_1(\mathbf{e}_1, \mathbf{e}_2) + c_2(\mathbf{e}_2, \mathbf{e}_2) + c_3(\mathbf{e}_3, \mathbf{e}_2) \\(\mathbf{f}, \mathbf{e}_3) &= c_1(\mathbf{e}_1, \mathbf{e}_3) + c_2(\mathbf{e}_2, \mathbf{e}_3) + c_3(\mathbf{e}_3, \mathbf{e}_3)\end{aligned}$$

We write the above three equations in the matrix form.

$$\begin{pmatrix} (\mathbf{e}_1, \mathbf{e}_1) & (\mathbf{e}_2, \mathbf{e}_1) & (\mathbf{e}_3, \mathbf{e}_1) \\ (\mathbf{e}_1, \mathbf{e}_2) & (\mathbf{e}_2, \mathbf{e}_2) & (\mathbf{e}_3, \mathbf{e}_2) \\ (\mathbf{e}_1, \mathbf{e}_3) & (\mathbf{e}_2, \mathbf{e}_3) & (\mathbf{e}_3, \mathbf{e}_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Notice that this is the same as the one obtained in Solution 1. So we omit the remaining calculation.

### Solution 3

A linear combination of  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  is written as  $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ . It is closest to  $f$  when it is the projection of  $f$  on the subspace spanned by  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ . That is, the vector  $f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3)$  is orthogonal to the subspace spanned by  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ . So the following three equations should hold.

$$\begin{aligned}(f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3), \mathbf{e}_1) &= 0 \\(f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3), \mathbf{e}_2) &= 0 \\(f - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3), \mathbf{e}_3) &= 0\end{aligned}$$

We expand the inner product in each of the equations.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_1) &= 0 \\(\mathbf{f}, \mathbf{e}_2) - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_2) &= 0 \\(\mathbf{f}, \mathbf{e}_3) - (c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \mathbf{e}_3) &= 0\end{aligned}$$

We move the second inner product into RHS in each of the equations.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) &= (c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3, \mathbf{e}_1) \\(\mathbf{f}, \mathbf{e}_2) &= (c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3, \mathbf{e}_2) \\(\mathbf{f}, \mathbf{e}_3) &= (c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + c_3 \mathbf{e}_3, \mathbf{e}_3)\end{aligned}$$

We expand the inner product in the RHS in each of the equations.

$$\begin{aligned}(\mathbf{f}, \mathbf{e}_1) &= c_1(\mathbf{e}_1, \mathbf{e}_1) + c_2(\mathbf{e}_2, \mathbf{e}_1) + c_3(\mathbf{e}_3, \mathbf{e}_1) \\(\mathbf{f}, \mathbf{e}_2) &= c_1(\mathbf{e}_1, \mathbf{e}_2) + c_2(\mathbf{e}_2, \mathbf{e}_2) + c_3(\mathbf{e}_3, \mathbf{e}_2) \\(\mathbf{f}, \mathbf{e}_3) &= c_1(\mathbf{e}_1, \mathbf{e}_3) + c_2(\mathbf{e}_2, \mathbf{e}_3) + c_3(\mathbf{e}_3, \mathbf{e}_3)\end{aligned}$$

We write the above three equations in the matrix form.

$$\begin{pmatrix} (\mathbf{e}_1, \mathbf{e}_1) & (\mathbf{e}_2, \mathbf{e}_1) & (\mathbf{e}_3, \mathbf{e}_1) \\ (\mathbf{e}_1, \mathbf{e}_2) & (\mathbf{e}_2, \mathbf{e}_2) & (\mathbf{e}_3, \mathbf{e}_2) \\ (\mathbf{e}_1, \mathbf{e}_3) & (\mathbf{e}_2, \mathbf{e}_3) & (\mathbf{e}_3, \mathbf{e}_3) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} (\mathbf{f}, \mathbf{e}_1) \\ (\mathbf{f}, \mathbf{e}_2) \\ (\mathbf{f}, \mathbf{e}_3) \end{pmatrix}$$

Notice that this is the same as the one obtained in Solution 1 and 2. So we omit the remaining calculation.