An introduction to Fourier transforms

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2015 July 14

This document is largely based on the reference book [1] with some parts slightly changed.

1 Fourier Integral

Suppose a periodic function $f_L(x)$ of period $2L$ is represented by a Fourier series

$$f_L(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos \omega_k x + b_k \sin \omega_k x)$$

where a_0, a_k, b_k, ω_k are given as follows.

$$\begin{aligned} \omega_k &= \frac{k\pi}{L} \\ a_0 &= \frac{1}{L}\int_{-L}^{L} f_L(x)\mathrm{d}x \\ a_k &= \frac{1}{L}\int_{-L}^{L} f_L(x)\cos \omega_k x \mathrm{d}x \\ b_k &= \frac{1}{L}\int_{-L}^{L} f_L(x)\sin \omega_k x \mathrm{d}x \end{aligned}$$

Then $f_L(x)$ is represented as follows.

$$\begin{aligned} f_L(x) &= \frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x + \frac{1}{L}\sum_{k=1}^{\infty}\left\{\cos \omega_k x \int_{-L}^{L} f_L(x)\cos \omega_k x \mathrm{d}x \right. \\ &\qquad \left. + \sin \omega_k x \int_{-L}^{L} f_L(x)\sin \omega_k x \mathrm{d}x\right\} \end{aligned}$$

We now set

$$\Delta\omega = \omega_{k+1} - \omega_k = \frac{(k+1)\pi}{L} - \frac{k\pi}{L} = \frac{\pi}{L}.$$

Since $1/L = \Delta\omega/\pi$ we may rewrite the above equation as follows.

$$\begin{aligned} f_L(x) &= \frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x + \frac{\Delta\omega}{\pi}\sum_{k=1}^{\infty}\left\{\cos\omega_k x\int_{-L}^{L} f_L(x)\cos\omega_k x\mathrm{d}x\right. \\ &\qquad \left. + \sin\omega_k x\int_{-L}^{L} f_L(x)\sin\omega_k x\mathrm{d}x\right\} \\ &= \frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x + \sum_{k=1}^{\infty}\left\{(\cos\omega_k x)\frac{1}{\pi}\int_{-L}^{L} f_L(x)\cos\omega_k x\mathrm{d}x\right. \\ &\qquad \left. +(\sin\omega_k x)\frac{1}{\pi}\int_{-L}^{L} f_L(x)\sin\omega_k x\mathrm{d}x\right\}\Delta\omega \end{aligned}$$

We now let $L \to \infty$ and assume that the resulting function

$$f(x) = \lim_{L\to\infty} f_L(x)$$

is *absolutely integrable* on the x-axis; that is, the following finite limits exist.

$$\lim_{a\to-\infty}\int_a^0 |f(x)|\mathrm{d}x + \lim_{a\to\infty}\int_0^a |f(x)|\mathrm{d}x$$

Note that this is written as $\int_{-\infty}^{\infty} |f(x)|\mathrm{d}x$. Then the first term $\frac{1}{2L}\int_{-L}^{L} f_L(x)\mathrm{d}x$ approaches zero. Also $\Delta\omega = \pi/L$ approaches zero and it seems *plausible* that the infinite series becomes an integral from 0 to ∞ as follows[1].

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left\{\cos\omega x\int_{-\infty}^{\infty} f(x)\cos\omega x\mathrm{d}x + \sin\omega x\int_{-\infty}^{\infty} f(x)\cos\omega x\mathrm{d}x\right\}\mathrm{d}\omega$$

By introducing the functions

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\mathrm{d}x \tag{1}$$

[1]Notice that I use the same name x for a few variables each of which has different scope. Of course we can rename inner x as other name such as v, which might be a usual way of writing this kind of formula.

and

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x \mathrm{d}x \tag{2}$$

we can also write the above formula as follows.

$$f(x) = \int_{0}^{\infty} \{A(\omega) \cos \omega x + B(\omega) \sin \omega x\} \, \mathrm{d}\omega \tag{3}$$

This is called a representation of $f(x)$ by a **Fourier integral**.

The following theorem holds (see p. 513 of the reference book [1]).

Theorem 1 *If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if $f(x)$ is absolutely integrable, then $f(x)$ can be represented by (3) with A and B given by (1) and (2). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point. In formula,*

$$\int_{0}^{\infty} \{A(\omega) \cos \omega x + B(\omega) \sin \omega x\} \, \mathrm{d}\omega = \frac{f(x-0) + f(x+0)}{2}.$$

□

Example Calculate the Fourier integral of the following function.

$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution We calculate $A(\omega)$ and $B(\omega)$ as follows.

$$\begin{aligned} A(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-1}^{1} \cos \omega x \mathrm{d}x \\ &= \frac{1}{\pi} \left[\frac{\sin \omega x}{\omega} \right]_{-1}^{1} \\ &= \frac{2 \sin \omega}{\pi \omega} \end{aligned}$$

$$\begin{aligned} B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x \mathrm{d}x \\ &= \frac{1}{\pi}\int_{-1}^{1} \sin\omega x \mathrm{d}x \\ &= 0 \end{aligned}$$

So we obtain the Fourier integral of $f(x)$ as follows.

$$\frac{2}{\pi}\int_0^{\infty} \frac{\cos\omega x \sin\omega}{\omega} \mathrm{d}\omega$$

By Theorem 1 we obtain the following equality.

$$\frac{2}{\pi}\int_0^{\infty} \frac{\cos\omega x \sin\omega}{\omega} \mathrm{d}\omega = \begin{cases} 1 & -1 < x < 1 \\ 1/2 & x = -1, 1 \\ 0 & \text{otherwise} \end{cases}$$

We should mention one thing. The integral above can be considered as the limit of the function

$$\frac{2}{\pi}\int_0^{a} \frac{\cos\omega x \sin\omega}{\omega} \mathrm{d}\omega$$

as a goes to infinity. In this integral, there are oscillations near the points $x = -1$ and $x = 1$. The oscillations does not disappear even if a increases, similarly to the Fourier series. This is called the Gibbs phenomenon.

Note By setting $x = 0$ in the Fourier integral of $f(x)$, we obtain the following equality.

$$\frac{2}{\pi}\int_0^{\infty} \frac{\sin\omega}{\omega} \mathrm{d}\omega = 1$$

By multiplying the both sides by $\frac{\pi}{2}$ we obtain the following equality.

$$\int_0^{\infty} \frac{\sin\omega}{\omega} \mathrm{d}\omega = \frac{\pi}{2}$$

This is called the **Dirichlet integral**. Consider the following (partial) integral (∞ is replaced by a) so called **sine integral**.

$$\mathrm{Si}(a) = \int_0^{a} \frac{\sin\omega}{\omega} \mathrm{d}\omega$$

In the sine integral $\mathrm{Si}(a)$, there are oscillations. The oscillations in the above integral come from the oscillations in the sine integral.

2 Fourier transform

The (real) Fourier integral is

$$f(x) = \int_0^\infty \{A(\omega)\cos\omega x + B(\omega)\sin\omega x\}\,\mathrm{d}\omega$$

where A and B are given as follows.

$$\begin{aligned}
A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v \mathrm{d}v \\
B(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v \mathrm{d}v
\end{aligned}$$

Substituting A and B into the integral, we have

$$\begin{aligned}
f(x) &= \int_0^\infty \left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\mathrm{d}v\cos\omega x + \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\mathrm{d}v\sin\omega x\right\}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(v)\cos\omega v\mathrm{d}v\cos\omega x + \int_{-\infty}^{\infty} f(v)\sin\omega v\mathrm{d}v\sin\omega x\right\}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(v)\cos\omega v\cos\omega x\mathrm{d}v + \int_{-\infty}^{\infty} f(v)\sin\omega v\sin\omega x\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(v)(\cos\omega v\cos\omega x + \sin\omega v\sin\omega x)\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(v)\cos(\omega x - \omega v)\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x - v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x - v))\mathrm{d}v\right\}\mathrm{d}\omega
\end{aligned}$$

(since the integral in the brackets is an even function of ω)

The integral of this form with sin instead of cos

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left\{\int_{-\infty}^{\infty} f(v)\sin(\omega(x - v))\mathrm{d}v\right\}\mathrm{d}\omega$$

is zero since the integral in the brackets is an odd function of ω.

By the Euler formula

$$e^{ix} = \cos x + i \sin x$$

we obtain the following equality.

$$e^{i\omega(x-v)} = \cos(\omega(x-v)) + i\sin(\omega(x-v))$$

By multiplying the both sides by $f(v)$ we obtain the following equality.

$$f(v)e^{i\omega(x-v)} = f(v)\cos(\omega(x-v)) + if(v)\sin(\omega(x-v))$$

By taking integral with respect to v and ω and multiplying the result by $\frac{1}{2\pi}$ we obtain the following equality.

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v)) + if(v)\sin(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&\quad +i\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\sin(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)\cos(\omega(x-v))\mathrm{d}v\right\}\mathrm{d}\omega \\
&= f(x)
\end{aligned}$$

So we obtain the following equality.

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega x - i\omega v}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{i\omega x}e^{-i\omega v}\mathrm{d}v\right\}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(v)e^{-i\omega v}\mathrm{d}v\right\}e^{i\omega x}\mathrm{d}\omega
\end{aligned}$$

We usually write this as follows[2].

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\mathrm{d}\omega \\
F(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x
\end{aligned}
$$

We call $\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\mathrm{d}x$ the Fourier transform of $f(x)$ and we call $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\mathrm{d}\omega$ the inverse Fourier transform of $F(\omega)$. Note that the above two equalities are just definitions of transformations and $f(x)$ might not be equal to the inverse Fourier transform of the Fourier transform of $f(x)$ (see Theorem 1).

Example Calculate the Fourier transform of the following function.

$$
f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}
$$

Solution

$$
\begin{aligned}
F(\omega) &= \int_{-1}^{1} e^{-i\omega x}\mathrm{d}x \\
&= \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-1}^{1} \\
&= \frac{1}{-i\omega}(e^{-i\omega} - e^{i\omega}) \\
&= \frac{1}{-i\omega}(\cos\omega - i\sin\omega - (\cos\omega + i\sin\omega)) \\
&= \frac{1}{-i\omega}(-2i\sin\omega) \\
&= 2\frac{\sin\omega}{\omega}
\end{aligned}
$$

[2]The constants ($\frac{1}{2\pi}$ and 1) depend on textbooks. They may be $\frac{1}{\sqrt{2\pi}}$ and $\frac{1}{\sqrt{2\pi}}$.

References

[1] Erwin Kreyszig. *Advanced Engineering Mathematics.* John Wiley & Sons Ltd., tenth edition, 2011.