Example 14

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Example Consider the periodic rectangular function $f_L(x)$ of period $2L > 2$ defined as follows.

$$f_L(x) = \begin{cases} 0 & -L < x < -1 \\ 1 & -1 < x < 1 \\ 0 & 1 < x < L \end{cases}$$

Calculate the Fourier series of $f_L(x)$ on the range $[-L, L]$ in the form of

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}\{a_k \cos \omega_k x + b_k \sin \omega_k x\}$$

where $\omega_k = \dfrac{k\pi}{L}$.

Solution Assume the following equation holds. (Note that there are no coefficients a_0, …, a_n that satisfy the equation, but it's ok.)

$$f_L(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\{a_k \cos \omega_k x + b_k \sin \omega_k x\}$$

Firstly integrate the both sides on $[-L, L]$.

$$\begin{aligned} \int_{-L}^{L} f_L(x)\mathrm{d}x &= \int_{-L}^{L} \frac{1}{2}a_0 \mathrm{d}x \\ &= La_0 \end{aligned}$$

So we obtain a_0 as follows.

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f_L(x)\mathrm{d}x \\ &= \frac{1}{L}\int_{-1}^{1} 1\mathrm{d}x \\ &= \frac{1}{L}\cdot 2 \\ &= \frac{2}{L} \end{aligned}$$

Secondly we multiply the both sides by $\cos\omega_k x$ and integrate them on the range $[-L, L]$.

$$\begin{aligned} \int_{-L}^{L} f_L(x)\cos\omega_k x\mathrm{d}x &= \int_{-L}^{L} a_k \cos^2\omega_k x\mathrm{d}x \\ &= La_k \end{aligned}$$

So we calculate a_k as follows.

$$\begin{aligned} a_k &= \frac{1}{L}\int_{-L}^{L} f_L(x)\cos\omega_k x\mathrm{d}x \\ &= \frac{1}{L}\int_{-1}^{1} \cos\omega_k x\mathrm{d}x \\ &= \frac{1}{L}\left[\frac{\sin\omega_k x}{\omega_k}\right]_{-1}^{1} \\ &= \frac{1}{L}\cdot\frac{\sin\omega_k - \sin(-\omega_k)}{\omega_k} \\ &= \frac{1}{L}\cdot\frac{\sin\omega_k + \sin\omega_k}{\omega_k} \\ &= \frac{1}{L}\cdot\frac{2\sin\omega_k}{\omega_k} \\ &= \frac{2}{L}\cdot\frac{\sin\omega_k}{\omega_k} \end{aligned}$$

Thirdly we multiply the both sides by $\sin\omega_k x$ and integrate them on the range $[-L, L]$.

$$\begin{aligned}\int_{-L}^{L} f_L(x) \sin \omega_k x \mathrm{d}x &= \int_{-L}^{L} b_k \sin^2 \omega_k x \mathrm{d}x \\ &= L b_k\end{aligned}$$

So we calculate b_k as follows.

$$\begin{aligned}b_k &= \frac{1}{L} \int_{-L}^{L} f_L(x) \sin \omega_k x \mathrm{d}x \\ &= \frac{1}{L} \int_{-1}^{1} \sin \omega_k x \mathrm{d}x \\ &= 0\end{aligned}$$

So the following is the linear combination closest to the function $f_L(x)$.

$$\frac{1}{L} + \sum_{k=1}^{n} \left\{ \frac{2}{L} \cdot \frac{\sin \omega_k}{\omega_k} \cos \omega_k x \right\}$$

The Fourier series is the limit of the linear combination as n goes to infinity.

$$\frac{1}{L} + \sum_{k=1}^{\infty} \left\{ \frac{2}{L} \cdot \frac{\sin \omega_k}{\omega_k} \cos \omega_k x \right\}$$

We depict the graphs of the relationship between ω_k and a_k for $L = 2$, 4, 8, 16, 32, and 64 in Fig. 1, 2, 3, 4, 5, and 6 respectively.

Comment

We obtain a non-periodic function by taking the limit of the function $f_L(x)$ as follows.

$$\lim_{L \to \infty} f_L(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The resulting function is a non-periodic function. This suggests a non-periodic function might be expanded (transformed) as well as a periodic one.

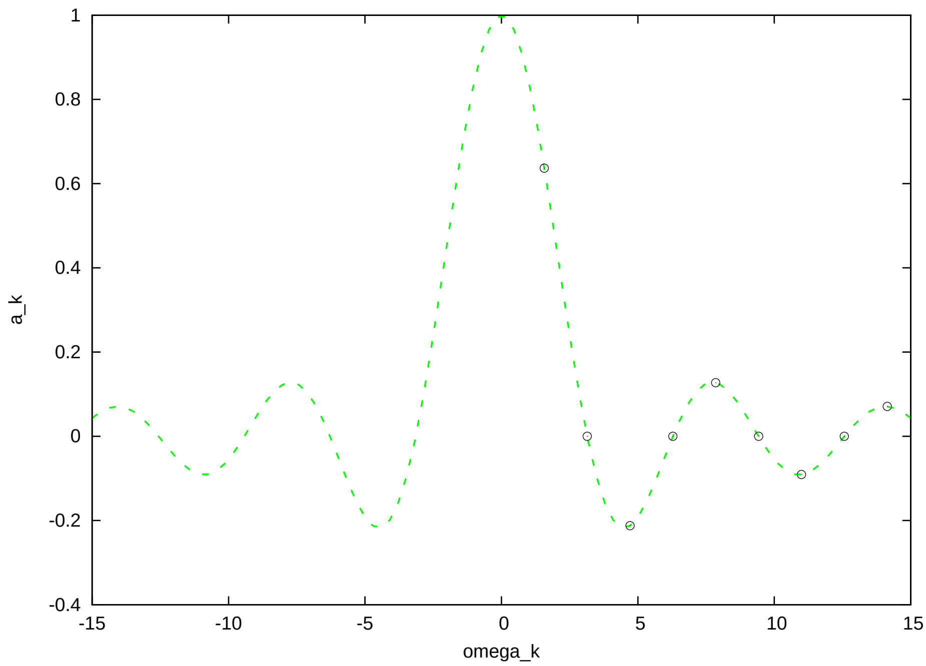

Figure 1: Relationship between ω_k and a_k for $L = 2$

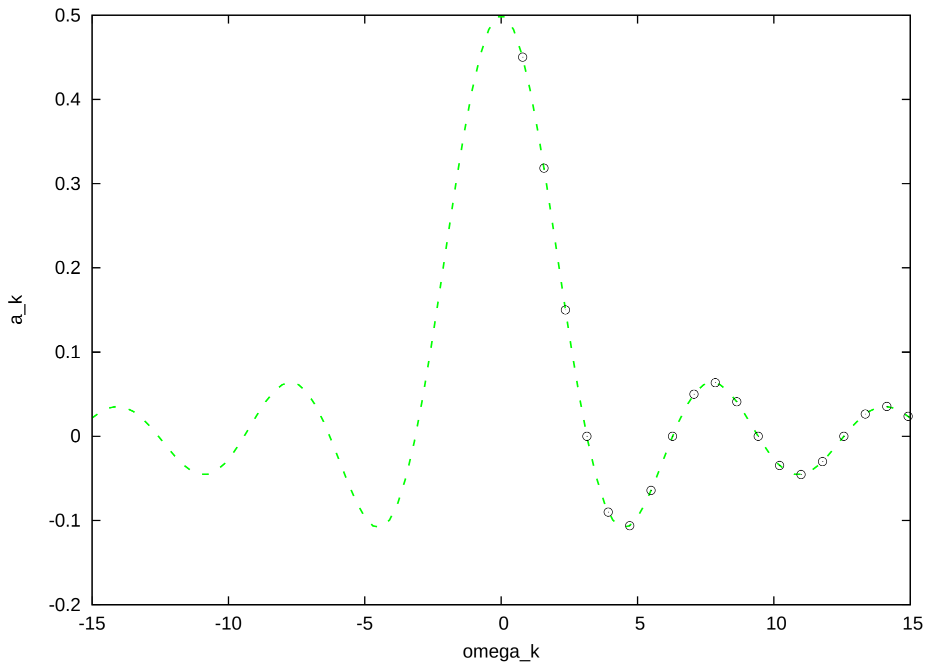

Figure 2: Relationship between ω_k and a_k for $L = 4$

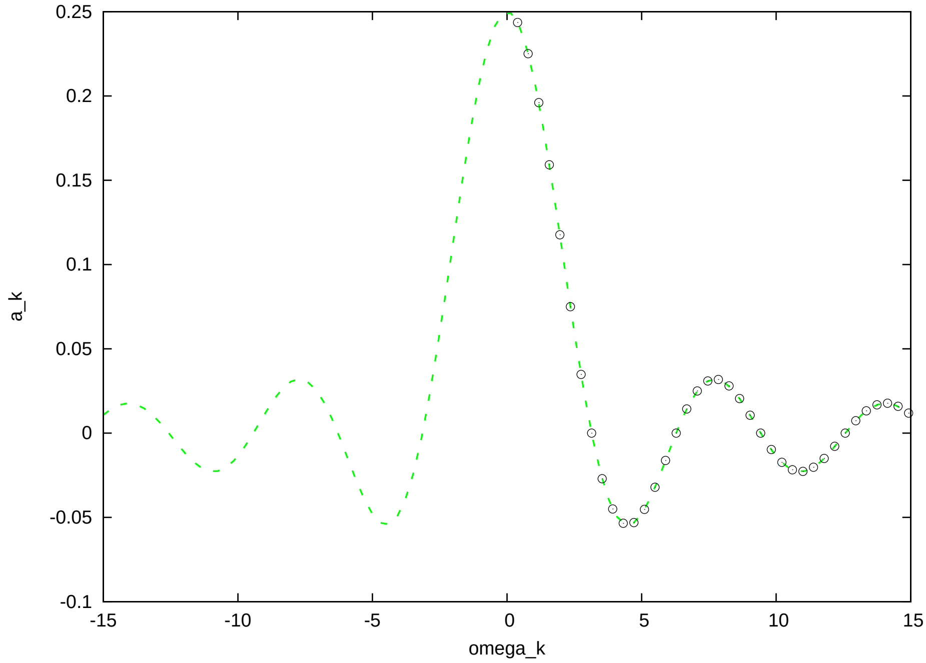

Figure 3: Relationship between ω_k and a_k for $L = 8$

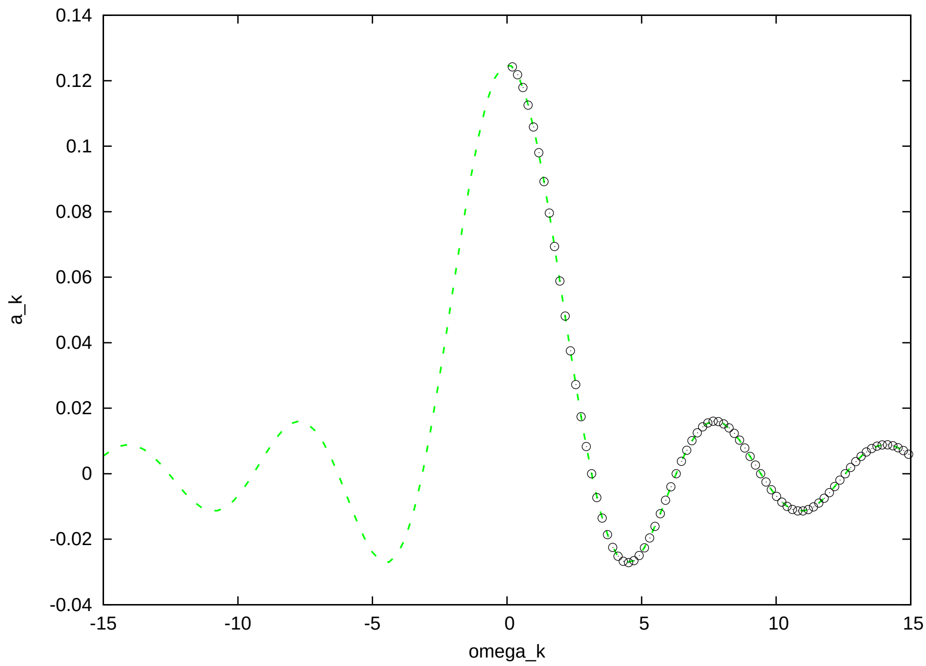

Figure 4: Relationship between ω_k and a_k for $L = 16$

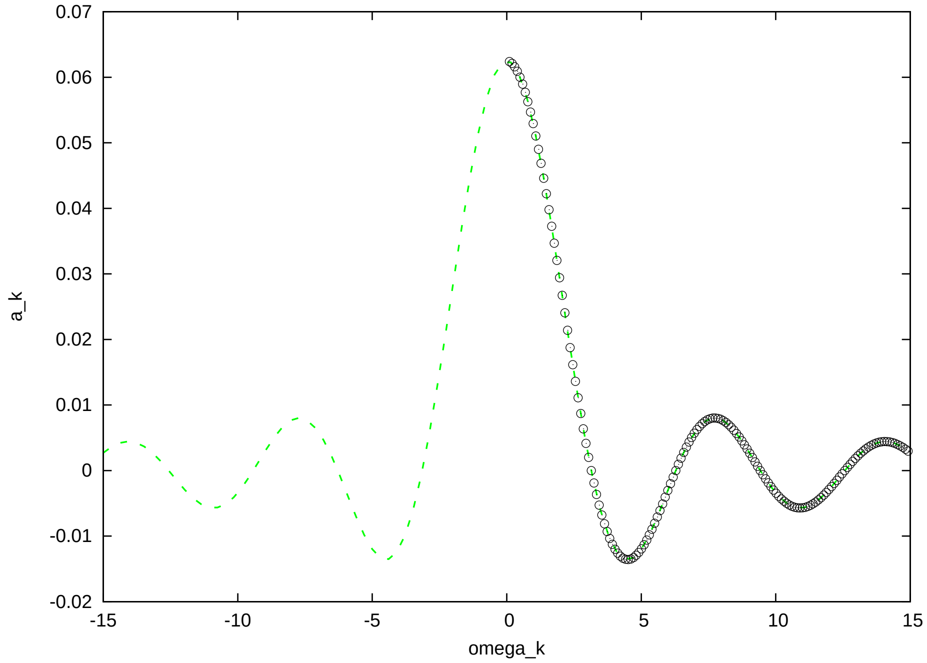

Figure 5: Relationship between ω_k and a_k for $L = 32$

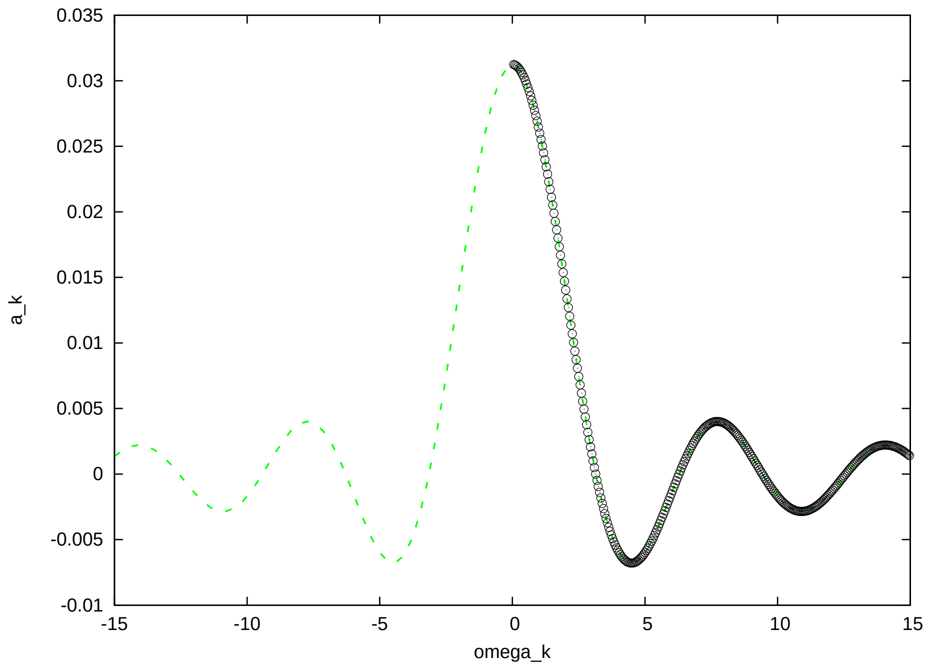

Figure 6: Relationship between ω_k and a_k for $L = 64$