

Exercise 7

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Exercise 7 Calculate $T_4(x)$.

Solution 1

$$\begin{aligned}\cos 4\theta &= \cos(3\theta + \theta) \\ &= \cos 3\theta \cos \theta - \sin 3\theta \sin \theta \\ &= (\cos 2\theta \cos \theta - \sin 2\theta \sin \theta) \cos \theta - (\sin 2\theta \cos \theta + \cos 2\theta \sin \theta) \sin \theta \\ &= \{(2 \cos^2 \theta - 1) \cos \theta - 2 \sin \theta \cos \theta \sin \theta\} \cos \theta \\ &\quad - \{2 \sin \theta \cos \theta \cos \theta + (2 \cos^2 \theta - 1) \sin \theta\} \sin \theta \\ &= \{(2 \cos^2 \theta - 1) \cos^2 \theta - 2 \sin^2 \theta \cos^2 \theta\} \\ &\quad - \{2 \sin^2 \theta \cos^2 \theta + (2 \cos^2 \theta - 1) \sin^2 \theta\} \\ &= \{2 \cos^4 \theta - \cos^2 \theta - 2(1 - \cos^2 \theta) \cos^2 \theta\} \\ &\quad - \{2(1 - \cos^2 \theta) \cos^2 \theta + (2 \cos^2 \theta - 1)(1 - \cos^2 \theta)\} \\ &= 2 \cos^4 \theta - \cos^2 \theta - 2 \cos^2 \theta + 2 \cos^4 \theta \\ &\quad - \{2 \cos^2 \theta - 2 \cos^4 \theta - 2 \cos^4 \theta + 3 \cos^2 \theta - 1\} \\ &= 4 \cos^4 \theta - 3 \cos^2 \theta - \{-4 \cos^4 \theta + 5 \cos^2 \theta - 1\} \\ &= 4 \cos^4 \theta - 3 \cos^2 \theta + 4 \cos^4 \theta - 5 \cos^2 \theta + 1 \\ &= 8 \cos^4 \theta - 8 \cos^2 \theta + 1\end{aligned}$$

Thus we obtain $T_4(x) = 8x^4 - 8x^2 + 1$.

Solution 2 By applying the recurrence formula of Chebyshev polynomials

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$$

to $T_3(x) = 4x^3 - 3x$ and $T_2(x) = 2x^2 - 1$, we obtain $T_4(x)$ as follows.

$$\begin{aligned}T_4(x) &= 2xT_3(x) - T_2(x) \\ &= 2x(4x^3 - 3x) - (2x^2 - 1) \\ &= 8x^4 - 6x^2 - 2x^2 + 1 \\ &= 8x^4 - 8x^2 + 1\end{aligned}$$