Two solutions for Exercise 4

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Exercise Approximate a column vector $\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ by a linear combination of the column vectors $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (i.e., $\sum_{k=1}^{2} c_k \boldsymbol{u}_k = c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2$ for some c_1 and c_2). That is, obtain c_1 and c_2 so that $c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2$ is closest to $\boldsymbol{a}$. As for the measure of the distance, use (the half of) the square of the norm of the difference of $c_1 \boldsymbol{u}_1 + c_2 \boldsymbol{u}_2$ and $\boldsymbol{a}$.

$$J = \frac{1}{2} \left\| \sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a} \right\|^2$$

The norm of a column vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is defined as follows.

$$\|\boldsymbol{x}\| = \sqrt{(\boldsymbol{x}, \boldsymbol{x})} = \sqrt{\sum_{k=1}^{3} x_k^2}$$

Solutions We show two solutions. One is by substituting the given column vectors into the normal equations and the other is by substituting them from the beginning. Solution 1 is clearer.

Solution 1 Firstly calculate J as follows.

$$\begin{aligned} J &= \frac{1}{2}\left\|\sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a}\right\|^2 \\ &= \frac{1}{2}\left(\sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a}, \sum_{k=1}^{2} c_k \boldsymbol{u}_k - \boldsymbol{a}\right) \\ &= \frac{1}{2}\left\{\left(\sum_{k=1}^{2} c_k \boldsymbol{u}_k, \sum_{k=1}^{2} c_k \boldsymbol{u}_k\right) - 2\left(\boldsymbol{a}, \sum_{k=1}^{2} c_k \boldsymbol{u}_k\right) + \|\boldsymbol{a}\|^2\right\} \\ &= \frac{1}{2}\left\{\sum_{k,l=1}^{2} c_k c_l (\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\sum_{k=1}^{2} c_k (\boldsymbol{a}, \boldsymbol{u}_k) + \|\boldsymbol{a}\|^2\right\} \end{aligned}$$

Partially differenciate this with recpect to c_i $(i = 1, 2)$.

$$\begin{aligned} \frac{\partial J}{\partial c_i} &= \frac{\partial}{\partial c_i}\frac{1}{2}\left\{\sum_{k,l=1}^{2} c_k c_l (\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\sum_{k=1}^{2} c_k (\boldsymbol{a}, \boldsymbol{u}_k) + \|\boldsymbol{a}\|^2\right\} \\ &= \frac{1}{2}\left\{\frac{\partial}{\partial c_i}\sum_{k,l=1}^{2} c_k c_l (\boldsymbol{u}_k, \boldsymbol{u}_l) - 2\frac{\partial}{\partial c_i}\sum_{k=1}^{2} c_k (\boldsymbol{a}, \boldsymbol{u}_k)\right\} \\ &= \frac{1}{2}\left\{2\sum_{k=1}^{2} c_k (\boldsymbol{u}_k, \boldsymbol{u}_i) - 2(\boldsymbol{a}, \boldsymbol{u}_i)\right\} \\ &= \sum_{k=1}^{2} c_k (\boldsymbol{u}_k, \boldsymbol{u}_i) - (\boldsymbol{a}, \boldsymbol{u}_i) \end{aligned}$$

By writing $\dfrac{\partial J}{\partial c_1} = 0$ and $\dfrac{\partial J}{\partial c_2} = 0$ in matrix form, we obtain

$$\begin{pmatrix} (\boldsymbol{u}_1, \boldsymbol{u}_1) & (\boldsymbol{u}_2, \boldsymbol{u}_1) \\ (\boldsymbol{u}_1, \boldsymbol{u}_2) & (\boldsymbol{u}_2, \boldsymbol{u}_2) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (\boldsymbol{a}, \boldsymbol{u}_1) \\ (\boldsymbol{a}, \boldsymbol{u}_2) \end{pmatrix}$$

By substituting column vectors $\boldsymbol{a}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$ in the above equation we obtain

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 3 \end{pmatrix}$$

By solving this we obtain

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

Thus the linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that is closest to the vector $\boldsymbol{a}$ is obtained as follows.

$$4\boldsymbol{u}_1 - \boldsymbol{u}_2 = 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$

Solution 2 By substituting $\boldsymbol{a}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$ in J we obtain

$$\begin{aligned}
J &= \frac{1}{2}\|c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 - \boldsymbol{a}\|^2 \\
&= \frac{1}{2}\left\| c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}\right\|^2 \\
&= \frac{1}{2}\left\| \begin{pmatrix} c_1 + c_2 - 3 \\ c_1 - 2 \\ c_1 - 6 \end{pmatrix}\right\|^2 \\
&= \frac{1}{2}\left\{c_1^2 + c_2^2 + 9 + 2c_1c_2 - 6c_1 - 6c_2 + c_1^2 - 4c_1 + 4 + c_1^2 - 12c_1 + 36\right\} \\
&= \frac{1}{2}\left\{3c_1^2 + c_2^2 + 2c_1c_2 - 22c_1 - 6c_2 + 49\right\}
\end{aligned}$$

Partially differenciate this with respect to c_1 and c_2.

$$\begin{aligned}
\frac{\partial J}{\partial c_1} &= \frac{1}{2}\{6c_1 + 2c_2 - 22\} = 3c_1 + c_2 - 11 \\
\frac{\partial J}{\partial c_2} &= \frac{1}{2}\{2c_1 + 2c_2 - 6\} = c_1 + c_2 - 3
\end{aligned}$$

Then we obtain the following systems of equations.

$$\begin{aligned}
3c_1 + c_2 &= 11 \\
c_1 + c_2 &= 3
\end{aligned}$$

By solving this we obtain $c_1 = 4, c_2 = -1$. Thus the linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that is closest to the vector $\boldsymbol{a}$ is obtained as follows.

$$4\boldsymbol{u}_1 - \boldsymbol{u}_2 = 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$$