The recurrence formula for Chebyshev polynomials

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May 26, 2015

Chebyshev polynomial $T_n(x)$ is obtained by substituting x for $\cos\theta$ in a formula which is obtained by expressing $\cos n\theta$ in a polynomial of $\cos\theta$. Hence the following equation holds.

$$T_n(\cos\theta) = \cos n\theta \qquad (n = 1, 2, \ldots)$$

By applying the addition theorem of cosine we obtain the following equation.

$$\begin{aligned} \cos(n+2)\theta &= \cos\{(n+1)\theta + \theta\} \\ &= \cos(n+1)\theta\cos\theta - \sin(n+1)\theta\sin\theta \end{aligned}$$

By applying the addition theorem of cosine we also obtain the following equation.

$$\begin{aligned} \cos n\theta &= \cos\{(n+1)\theta - \theta\} \\ &= \cos(n+1)\theta\cos\theta + \sin(n+1)\theta\sin\theta \end{aligned}$$

By adding these two equations we obtain

$$\cos(n+2)\theta + \cos n\theta = 2\cos\theta\cos(n+1)\theta.$$

So we obtain

$$\cos(n+2)\theta = 2\cos\theta\cos(n+1)\theta - \cos n\theta$$

and hence the following formula.

$$T_{n+2}(x) = 2T_1(x)T_{n+1}(x) - T_n(x)$$

Since $T_1(x) = x$, we obtain the following recurrence formula.

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$$

An example We calculate $T_2(x)$ by applying the above recurrence formula to $T_1(x) = x$ and $T_0(x) = 1$.

$$\begin{aligned} T_2(x) &= 2xT_1(x) - T_0(x) \\ &= 2x^2 - 1 \end{aligned}$$

This coincides with the result obtained from $\cos 2\theta$.

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \end{aligned}$$