Set Theory and Infinity

November 18th (Mon) – 22nd (Fri)

Last update: November 14th, 2019

Program

	Monday	Tuesday	Wednesday	Thursday	Friday
9:30 | 10:30	Karagila	Karagila	Karagila	Karagila	Karagila
	break	break	break	break	break
11:00 | 11:30	Eda	Adolf	break	Asperó	Cardona
11:30 | 12:00	Sobota	Du	break	Switzer	Yorioka
	break	break	Excursion	break	break
13:30 | 14:00	Schrittesser	Schindler		Velickovic	Kada
14:00 | 14:30	Bagaria	Schindler		Velickovic	Schlicht
	break	break		break	break
15:00 | 15:30	Eskew	Schindler		Velickovic	Ottenbreit
15:30 | 16:00	Williams	Schindler		Velickovic	Cox
	break	break		break	break
16:15 | 16:45	Usuba	Mejía		Miyamoto	Fuchino

Abstracts

1 Tutorial

Asaf Karagila: The Bristol Model

Let V be a model of ZFC, and let $V[G]$ be a [set-]generic extension of V. What can we say about intermediate models $V \subseteq W \subseteq V[G]$? Well, if they are models of ZFC as well, we know perfectly well what type of models we can find, those are all intermediate forcing extensions given by complete subforcings, and G is always generic over W.

What happens if we assume only ZF holds, though? Well, it turns out that there is a lot more to explore in that multiverse. First are the symmetric extensions, intermediate models satisfying ZF and defined using automorphisms of the forcing, but are there more? In 2011, in Bristol, a small workshop was held about Woodin's HOD Conjecture, and there they showed that taking $V = L$ and G to define a single Cohen real, c, we can in fact define an intermediate model which is not constructible from any set. In other words, there is an intermediate model $L \subseteq M \subseteq L[c]$ such that $M \neq L(x)$ for all x.

This work was not written down by the participants of the workshop, but was later reworked into [1] using a novel technique for iterating symmetric ex-tensions. This allowed us to closely investigate both the necessary assumptions taken from $V = L$, as well as further implications of this work.

In this tutorial we will discuss what are symmetric extensions, how to iterate them, what are the tools necessary for the construction of the Bristol model, and some of the implications its existence has on the generic multiverse of modelsof ZF. More importantly, we will not assume prior knowledge about these techniques (beyond forcing and the basics of set theory).

References

[1] Asaf Karagila, *The Bristol model: an abyss called a Cohen real*, J. Math. Log. **18** (2018), no. 2, 1850008, 37. MR 3878470

2 Invited talks

Ralf Schindler: MM^{++} implies (*)

We show that Martin's Maximum^{++} implies Woodin's $\mathbb{P}_{\max}$ axiom (*). This is joint work with David Asperó.

Boban Velickovic: In search of higher forcing axioms

What form should higher forcing axioms have? I will discuss some limitations in the attempts to generalize PFA and other strong forcing axioms to $\aleph_2$. Then I will present an iteration scheme using virtual models as side conditions that can be used to iterate certain forcing notions while preserving two cardinals. As an application of this method, I will present a principle that has many of the desired consequences of higher forcing axioms, such as the tree property at $\aleph_2$ and $\aleph_3$, the approachability ideal on $\aleph_2$ being the non stationary ideal on cofinality ω_1, the failure of square, SCH, etc.

This is part of an ongoing project with my PhD students Rahman Mohammadpour.

3 Contributed talks

Dominik Adolf: Core Model Induction, Jónsson cardinals and long Chang's conjectures.

In a recent result the speaker has shown that the existence of a "small" Jónsson cardinal implies the existence of a sharp for a model with a strong cardinal (an improvement on earlier work joint with S. Cox and P. Welch). Since then the speaker has noticed that the same argument also applies to certain forms of Chang's conjecture, e.g. $(\aleph_{\omega+\omega}, \aleph_\omega) \twoheadrightarrow (\aleph_\omega, \aleph_n)$. This is significant as instances of these are known to be consistent.

Here we will introduce a new result (joint with D. Saatrup-Nielsen) that shows the existence of an inner model with a Woodin cardinal from such a property with some mild extra assumptions. This means that these properties could also allow us to apply Core Model Induction. We will discuss some of the difficulties that arise when using CMI in this context. We hope that work done in this context might also apply to more commonly discussed forms of Chang's Conjecture.

David Asperó: Extensions of Martin's Axiom, $\square_{\omega_1}$, and a forcing axiom failure.

I will present consistent and inconsistent forcing axioms extending Martin's Axiom, and will prove that one such (consistent) forcing axiom implies square at ω_2.

Joan Bagaria: The Weak Vopenka Principle for definable classes of structures

Trevor Wilson showed recently that the Weak Vopenka principle is equivalent to the assertion that the class ORD of ordinals is Woodin, thus solving in a very unexpected way a decades-long open problem. In a joint work, we prove level-by-level forms of this result and show, e.g., that the Weak Vopenka Principle for classes of relational structures that are Σ_2 definable is equivalent to the existence of a strong cardinal.

Miguel A. Cardona: New consistency results about cardinal invariants associated with the strong measure zero ideal

Yorioka [3] constructed a *matrix* of subsets of the reals, which gives a Tukey isomorphism between the σ-ideal of stong measure zero sets $\mathcal{SN}$ and $\langle \kappa^\kappa, \leq^* \rangle$, to prove that $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_\kappa$ (the *dominating number on* κ^κ) whenever $\mathrm{add}(\mathcal{I}_f) = \mathrm{cof}(\mathcal{I}_f) = \kappa$ for all increasing f (the $\mathcal{I}_f$ are the *Yorioka ideals*).

In this talk we introduce a *complete dominating system* (see [1]) that generalizes Yorioka's matrix in some sense, and we construct a *complete dominating system* via a forcing matrix iterations of ccc posets to force

(I) $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) < \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.

(II) $\mathrm{add}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$.

On the other hand, the speaker with Mejía and Rivera-Madrid [2] showed that, in Sacks model, $\mathrm{non}(\mathcal{SN}) < \mathrm{cov}(\mathcal{SN}) < \mathrm{cof}(\mathcal{SN})$. These are first results where 3 cardinal invariants associated with $\mathcal{SN}$ are pairwise different

References

[1] Cardona, Miguel A., *About cardinals characteristcs associated with the strong measure zero ideal*, in preparation.

[2] Cardona, Miguel A. and Mejía, Diego A. and Ismael Rivera-Madrid, *The covering number of thestrong measure zero ideal can be above almost everything else*, arXiv:1902.01508.

[3] T. Yorioka, *The cofinality of the strong measure zero ideal*, J. Symb.Logic 67 (2002) 1373-1384.

Sean Cox: Martin's Maximum and reflection properties of Stationary Logic

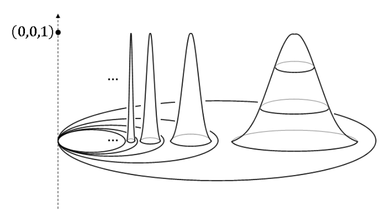
The Diagonal Reflection Principle (DRP) is a maximal form of simultaneous stationary set reflection, which is roughly equivalent (by recent work of Fuchino-Rodrigues-Sakai) to the assertion that Stationary Logic has the Downward Loewenheim-Skolem-Tarski Property (i.e. that every structure has an elementary substructure, with respect to Stationary Logic, of size $< \omega_2$). DRP follows from the ++ version of the forcing axiom for σ-closed posets; in particular, MM^{++} implies DRP. I will discuss the proof (joint with Hiroshi Sakai) that Martin's Maximum (or even $\mathrm{MM}^{+\omega}$) does not imply DRP.

Jin Du: Squares and Uncountably Singularized Cardinals

I will discuss a recent paper by Levine and Sinapova. They showed how to find a model V with κ inaccessible and a forcing extension W such that κ^+ remains a cardinal, κ is singularized to have uncountable cofinality in W, and $\square_{\kappa,\tau}$ fails for all $\tau < \kappa$. This is in contrast to the countable cofinality case, where $\square_{\kappa,\omega}$ holds.

Katsuya Eda: Question on Archipelago groups

The classical archipelago group is a quotient group of the fundamental group of the Hawaiian earring by the normal closure of the free group of countable rank, which is denoted by $\mathcal{A}(\mathbb{Z})$. This group is isomomorphic to the fundamental group of the harmonic archipelago. In the figure there are mountains of the same height and the areas of islands converge to zero and the shore lines of islands converge to one point of the sea. We consider the surface consisting of the mountains and the sea as a space. We remark that the surface of each mountain is homeomorphic to a disk and is ignored in the fundamental group.



To generalize this construction we regard the surface of a mountain as a cone over a circle. For given path connected spaces X_i, we connect X_i and X_{i+1} by an interval I_i of length $1/i$ and put a point o to which X_i and I_i converge. Then we extend X_i to a cone CX_i so that some neighborhood of the point o does not contain CX_i for any i. This space is the archipelago $\mathcal{A}(X_i : i < \omega)$. Now the fundamental group $\pi_1(\mathcal{A}(X_i : i < \omega))$ is the archipelago group. Some hearers may not know the fundamental groups of spaces, but he does not need to learn it, since we start from the presentation of this group and the questions here are really group theoretic ones. We have introduced achipelagos to ensure that the aripelago groups are natural objects and deserve to be studied.

Since the fundamental group of the Hawaiian earring is expressed by the free σ-product $\circledast_\omega \mathbb{Z}$, we obtain an archipelago group $\mathcal{A}(G)$ by replacing $\mathbb{Z}$ with G. In [1] the authors asserted that $\mathcal{A}(\mathbb{Z})$ and $\mathcal{A}(\mathbb{Z}/k\mathbb{Z})$ are isomorphic for $k \geq 3$. But there is a gap in their proof and now the main question is

Are $\mathcal{A}(\mathbb{Z})$ and $\mathcal{A}(\mathbb{Z}/k\mathbb{Z})$'s for $k \geq 2$ isomorphic or not?

Below we state the definition of free σ-products with a basic result [2].

Definition 3.1. Let $G_i (i \in I)$ be groups. We assume $G_i \cap G_j = \{e\}$ for distinct $i, j \in I$. Elements of $\bigcup_{i\in I} G_i \setminus \{e\}$ are called *letters*. W is a word, if W is a map from a linearly ordered set $\overline{W}$ to $\bigcup_{i\in I} G_i \setminus \{e\}$ such that $W^{-1}(G_i)$ is finite for each i. In case the cardinality of $\overline{W}$ is countable, we say that W is a σ-word. The class of all words is denoted by $\mathcal{W}(G_i : i \in I)$ and the class of all σ-words is denoted by $\mathcal{W}^\sigma(G_i : i \in I)$.

If there exists an isomorphism $i : \overline{U} \to \overline{V}$ as linearly ordered sets and $U(\alpha) = V(i(\alpha))$ for all $\alpha \in \overline{U}$, we say that U and V are isomorphic and denote it by $U \equiv V$ and we identify U and V. A word U is a *subword* of a word W, if $W \equiv V_0 U V_1$ for some words V_0, V_1. A word W is *reduced*, if $U \neq e$ for any non-empty subword U of W.

Proposition 3.2. *[2, Theorem 1.4] For each word W there exists a unique reduced word W_0 such that $W_0 = W$.*

For finite subsets X and Y of an index set I satisfying $X \subseteq Y$, let $p_{XY} : *_{i\in Y} G_i \to *_{i\in X} G_i$ be the projection. The inverse limit $\varprojlim(G_i, p_{XY} : i \in I, X \subseteq Y \Subset I)$ is called the unrestricted free product [3]. Since a word $W \in \mathcal{W}(G_i : i \in I)$ can be regarded as an element of $\varprojlim(G_i, p_{XY} : i \in I, X \subseteq Y \Subset I)$, we write $U = V$ when they are equal in the unrestricted free product for $U, V \in \mathcal{W}(G_i : i \in I)$.

Define $\circledast_{i<\omega} G_i$ and $\circledast^\sigma_{i<\omega} G_i$ to be the subgroups of the unrestricted product consisting of elements expressed by words in $\mathcal{W}(G_i : i \in I)$ and $\mathcal{W}^\sigma(G_i : i \in I)$ respectively.

Next we define archipelago groups. For given groups G_i $(i < \omega)$, define $\mathcal{A}(G_i : i < \omega)$ to be the quotient group of the free σ-product $\circledast_{i<\omega} G_i$ factored by $N(*_{i<\omega} G_i)$, which is the normal closure of the free product $*_{i<\omega} G_i$.

The following canonical homomorphisms are relevant $\sigma_G : \circledast_{i<\omega} G_i \to \circledast_{i<\omega} G_i / N(*_{i<\omega} G_i)$ and $\sigma_H : \circledast_{i<\omega} H_i \to \circledast_{i<\omega} H_i / N(*_{i<\omega} H_i)$ be the quotient homomorphisms.

References

[1] Gregory R. Conner, Wolfram Hojka, and Mark Meilstrup, *Archipelago groups*, Proc. Amer. Math. Soc. **143** (2015), no. 11, 4973–4988. MR 3391054

[2] Katsuya Eda, *Free σ-products and non-commutatively slender groups*, J. Algebra **148** (1992), no. 1, 243–263. MR 1161575

[3] Graham Higman, *Unrestricted free products, and varieties of topological groups*, J. London Math. Soc. **27** (1952), 73–81. MR 45730

Monroe Eskew: Categories of amenable embeddings

An elementary embedding $j : M \to N$ between transitive models of set theory is called amenable if the pointwise image of every ordinal in M is an element of N. We consider the category $\mathcal{A}_\delta$ whose objects are all transitive models of ZFC of a fixed height δ, and whose arrows are amenable embeddings. We give some partial information about the structure of these categories, showing under large cardinals that there is a countable δ such that $\mathcal{A}_\delta$ contains subcategories isomorphic to several canonical linear and partial orders, including the real line. We conclude with some open questions. This is joint work with Sy Friedman.

Sakaé Fuchino: On a/the solution of the continuum problem

Strong reflection principles with the reflection cardinal $\leq \aleph_1$ or $< 2^{\aleph_0}$ imply that the size of the continuum is either $\aleph_1$ or $\aleph_2$ or very large. Thus, the assumption, that a strong reflection principle with these reflection cardinals should hold, seems to support the trichotomy of the possible size of the continuum. We consider generic large cardinal properties which imply these reflection principles and some other combinatorial properties.

Masaru Kada: Hat guessing games with many colors

We discuss "puzzles of prisoners and hats" with infinitely many prisoners and more than two hat colors.

Assuming that the set of hat colors is equipped with a commutative group structure, we prove strategic equivalence among puzzles of several protocols with countably many prisoners.

Diego Mejía: Cichoń's maximum without large cardinals

Is it consistent with ZFC that all the (non-dependant) cardinals in Cichoń's diagram are pairwise different? A couple of years ago, Goldstern, Kellner and Shelah used large cardinals to answer this question affirmatively. Just very recently, the speaker joined their work, and they managed to show how to prove this consistency result without using large cardinals. In this talk, the main ideas and a sketch of the proof will be presented.

Tadatoshi Miyamoto: Forcing continuous epsilon-chains with finite side conditions

We force a continuously inclusion-ship increasing epsilon-chain of countable sets that are ground-model-elementary substructures of a relevant ground-model-relational structure. The least uncountable cardinal is the length of this chain. The conditions are finite epsilon-chains that have associated fast functions.

As an example of this type of method, we provide a single semi-proper step for forcing the Strong Reflection Principle (SRP). Hence under, for example, the Martin's Maximum (MM), we attain SRP by this finite side condition method.

We do not know real differences between this finite method and a usual countable one. We have not exploited finite symmetric cases, yet.

André Ottenbreit Maschio Rodrigues: Some reflection principles at large continuum and an application of mixed support iteration

This is a joint work with Sakaé Fuchino and Hiroshi Sakai.

Given some weak second order logic $\mathcal{L}$ and a regular cardinal κ, we define the Strong Downward Löwenheim-Skolem reflection on $\mathcal{L}$ down to $< \kappa$ (denoted by $\mathrm{SDLS}(\mathcal{L}, < \kappa)$) as the statement:

> For any structure $\mathfrak{A}$ of countable signature and cardinality $\geq \kappa$, there is a structure $\mathfrak{B}$ of cardinality $< \kappa$ such that $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

In this talk, we discuss variations of this SDLS reflection principle related to stationary logics which shall be defined by introducing the second order quantifier $\mathrm{stat}\, X$, interpreted as "for stationary many X". We discuss the relationship between these statements and other reflection principles such as the Fodor-Type Reflection Principle (FRP), Diagonal Reflection Principle (DRP), Game Reflection Principle (GRP).

Then, we use a mixed support iteration to construct, starting from a model with 2 supercompact cardinals, a model where the continuum is fairly large, some fragment of MA holds (enough to make $\mathsf{add}(\text{null}) = 2^{\aleph_0}$) and a strong version of SLDS down to $\leq 2^{\aleph_0}$ holds simultaneously with a weaker version of SDLS down to $< 2^{\aleph_0}$.

Philipp Schlicht: A variant of generic supercompactness

We consider a natural variant of generically supercompact cardinals, with elementary embeddings of arbitrarily large H_θ instead of V that exist in generic extensions. The exact strength of these cardinals is not yet known, but it follows from a result of Usuba that they are weaker than some large cardinals that can exist in L. We study their indestructibility properties, for instance we show that they are always indestructible under $<\kappa$-directed closed forcing. This is joint work in progress with Dan Nielsen.

David Schrittesser: TBA

Damian Sobota: Set-theoretic aspects of convergence of sequences of measures

During my talk we will study various set-theoretic aspects of convergence of sequences of finitely additive signed measures on Boolean algebras—we will see, e.g., what impact inequalities between cardinal characteristics of the continuum have on properties of Boolean algebras concerning different types of convergence of measures, learn some theorems regarding preservations of those properties by forcing extensions and see what the special place of minimally generated Boolean algebras in this topic is.

Corey Bacal Switzer: ∞-subversion forcing

The classes of subcomplete and subproper forcing were introduced by Jensen during, amongst other things, his study of the extended Namba problem. These classes generalize the class of σ-closed and proper forcing notions respectively, are iterable with revised countable support and have many interesting properties. Subcomplete forcing in particular is very interesting as it adds no reals but may add ω-cofinal sequences to larger cardinals. The corresponding axiom SCFA is compatible with CH, and even $\diamondsuit$, but still implies many of the same consequences as MM, including the failure of several square principles.

In an effort to understand better models of SCFA, and the role of the size of the continuum in them, we discovered even more general versions of subcomplete and subproper forcing. These classes are called ∞-subcomplete and ∞-subproper forcing. Both of these "∞-subversion" forcing classes are iterable with nice iterations in the sense of Miyamoto. In this talk I will introduce ∞-subversion forcing and discuss their iteration theorems, as well as some related preservation theorems. Finally I will give some applications to the study of forcing axioms compatible with CH. This is joint work with Gunter Fuchs.

Toshimichi Usuba: Geology of symmetric grounds

A symmetric extension, which is a submodel of a generic extension, is frequently used to construct a choiceless model. Let us say that an inner model is symmetric ground if the universe is a symmetric extension of it. In this talk, we study set-theoretic geology of symmetric grounds, that is, the structure of all symmetric grounds.

Kameryn J Williams: On axioms for multiverses of set theory

The multiversist position in the philosophy of set theory holds that rather than there being an absolute notion of set, there are instead many universes of set theory, each of equal ontological status. Hamkins proposed a series of Multiverse Axioms to capture his position on the structure of the set theoretic multiverse, with his Well-Foundedness Mirage axiom being the most provocative. Gitman and Hamkins showed that the collection of countable, recursively saturated models of set theory form a multiverse satisfying Hamkins's Multiverse Axioms. And it is easy to check that if a multiverse of models of set theory satisfies the Well-Foundedness Mirage axiom then every world in that multiverse must be recursively saturated.

Gitman, Godziszewski, Meadows, and I investigated whether this forced recursive saturation could be avoided by weakening the Well-Foundedness Mirage axiom. We considered two weakenings, and showed that neither of them forces all worlds in a multiverse to be recursively saturated. In this talk I will discuss the construction of our two multiverses.

Teruyuki Yorioka: Todorcevic's fragments of Martin's Axiom and some weak form of uniformization of a ladder system coloring

Uniformization of ladder system colorings has been introduced by analysis of a proof of the Shelah's solution of Whitehead problem. Here, for a subset $\mathcal{S}$ of the powerset of $\omega_1 \cap \mathsf{Lim}$, $\mathsf{U}(\mathcal{S})$ is the assertion that, for any ladder system coloring $\langle d_\alpha : \alpha \in \omega_1 \cap \mathsf{Lim} \rangle$, there exists $S \in \mathcal{S}$ such that the restricted coloring $\langle d_\alpha : \alpha \in S \rangle$ can be uniformized. Shelah's proof can be separated the following two theorems: $\mathsf{MA}_{\aleph_1}$ implies $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim})$, and $\mathsf{U}(\{\omega_1 \cap \mathsf{Lim})$ implies the existence of a non-free Whitehead group. We argue relationships between Todorcevic's fragments of Martin's Axiom and our weak form $\mathsf{U}(\mathcal{S})$ of a ladder system coloring. Especially, we focus on the assertion $\mathsf{U}(\mathsf{stat})$.