Forcing absoluteness and regularity properties

Daisuke Ikegami (Universiteit van Amsterdam)

May 27th, 2009
Work in ZFC

Models $= \text{transitive models of ZFC}$

$V$: the class of all sets
Forcing absoluteness

**Definition**

$M$: a model, $\mathbb{P} \in M$: a partial order, $\Gamma$: a set of statements

$M$ is $\Gamma$-$\mathbb{P}$-absolute if

\[ \forall \phi \in \Gamma, \forall G: \text{a } \mathbb{P}\text{-generic filter over } M, \]

\[ M \models \phi \iff M[G] \models \phi. \]

- In this talk, $\Gamma$ will be $\Sigma^1_n$ for some $n \geq 1$.
- For $\Sigma^1_n$ statements, we allow reals in $M$ as parameters.
Remark

- $V$ is $\Sigma^1_2$-$\mathbb{P}$-absolute for any $\mathbb{P}$.
- $L$ is not $\Sigma^1_3$-$\mathbb{P}$-absolute if $\mathbb{P}$ adds a new real.
- $\text{MA}_{\aleph_1}$ implies that “$V$ is $\Sigma^1_3$-$\mathbb{P}$-absolute for any ccc $\mathbb{P}$”.
Examples: the Baire property, Lebesgue measurability

Remark

- Every $\Sigma^1_1$-set of reals has the Baire property and is Lebesgue measurable.
- In L, there is a $\Delta^1_2$-set of reals without the Baire property and which is not Lebesgue measurable.
- $MA_{\aleph_1}$ implies that “Every $\Delta^1_2$-set of reals has the Baire property and is Lebesgue measurable”.
Connection between forcing absoluteness and regularity properties

**Theorem (Bagaria, Judah-Shelah, Woodin)**

\[ C: \] Cohen forcing, the following are equivalent:

1. \( V \) is \( \Sigma^1_3 \)-absolute,
2. every \( \Delta^1_2 \)-set of reals has the Baire property,
3. for any real \( a \), there is a Cohen real over \( L[a] \).
Connection between forcing absoluteness and regularity properties

**Theorem (Bagaria, Judah-Shelah, Woodin)**

\( \mathcal{C} \): Cohen forcing, the following are equivalent:

1. \( V \) is \( \Sigma^1_3 \)-absolutely Cohen,
2. every \( \Delta^1_2 \)-set of reals has the Baire property,
3. for any real \( a \), there is a Cohen real over \( L[a] \).

**Theorem (Bagaria, Judah-Shelah, Woodin)**

\( \mathcal{B} \): random forcing, the following are equivalent:

1. \( V \) is \( \Sigma^1_3 \)-absolutely random,
2. every \( \Delta^1_2 \)-set of reals is Lebesgue measurable,
3. for any real \( a \), there is a random real over \( L[a] \).
Connection between forcing absoluteness and regularity properties 2

**Theorem (Brendle-Löwe, I.)**

$\mathcal{S}$: Sacks forcing, the following are equivalent:

1. $V$ is $\Sigma^1_3$-$\mathcal{S}$-absolute,
2. there is no $\Delta^1_2$ Bernstein set, and
3. for any real $a$, there is a real $x$ which is not in $L[a]$.

**Remark**

We **cannot** replace 3. by the following:
For any real $a$, there is a Sacks real $s$ over $L[a]$.
More examples

The same kind of statement holds for

- Mathias forcing and Ramsey property,
- Hechler forcing and the Baire property for Dominating topology.
Goal: introduce regularity properties for a wide class of forcings and prove the same kind of results in a uniform way.
Generalization

Goal: introduce regularity properties for a wide class of forcings and prove the same kind of results in a uniform way.

**Theorem (1.)**

$\mathbb{P}$: proper, strongly arboreal

Then the following are equivalent:

1. $V$ is $\Sigma^1_3$-$\mathbb{P}$-absolute,
2. every $\Delta^1_2$-set of reals is $\mathbb{P}$-measurable.
Theorem (1.)

$\mathbb{P}$: proper, strongly arboreal, provably $\Delta^1_2$. Assume

$$\{c \mid c : \text{a Borel code}, B_c \in l_{\mathbb{P}^*}\} \in \Sigma^1_2. \ (\ast)$$

Then the following are equivalent:

1. $V$ is $\Sigma^1_3$-$\mathbb{P}$-absolute,
2. every $\Delta^1_2$-set of reals is $\mathbb{P}$-measurable,
3. for any real $a$ and any $T \in \mathbb{P}$, there is a quasi-$\mathbb{P}$-generic real $x$ in $[T]$ over $L[a]$. 
Definable forcings and strongly proper forcings

**Definition**

\( \mathbb{P} \): a partial order, \( n \geq 1 \)

1. \( \mathbb{P} \) is *provably \( \Delta^1_n \)* if there are \( \Sigma^1_n \)-formula and \( \Pi^1_n \)-formula such that ZFC proves that they both define the triple \((\mathbb{P}, \leq_\mathbb{P}, \perp_\mathbb{P})\).

2. \( \mathbb{P} \) is *proper* if for any countable elementary submodel \( M \) of \( \mathcal{H}_\theta \) for enough large \( \theta \) with \( \mathbb{P} \in M \) and for any \( p \in \mathbb{P} \cap M \), there is a \( q \leq p \) s.t. \( q \) is \((M, \mathbb{P})\)-generic (i.e. for any maximal antichain \( A \) of \( \mathbb{P} \) in \( M \), \( A \) is predense below \( q \)).
Definable forcings and strongly proper forcings

Definition

\( \mathbb{P} \): a partial order, \( n \geq 1 \)

1. \( \mathbb{P} \) is provably \( \Delta^1_n \) if there are \( \Sigma^1_n \)-formula and \( \Pi^1_n \)-formula such that ZFC proves that they both define the triple \( (\mathbb{P}, \leq, \bot_\mathbb{P}) \).

2. \( \mathbb{P} \) is strongly proper if for any countable transitive model \( M \) with \( (\mathbb{P}^M, \leq^M, \bot^M) \subseteq (\mathbb{P}, \leq, \bot) \) and for any \( p \in \mathbb{P}^M \), there is a \( q \leq p \) s.t. \( q \) is \( (M, \mathbb{P}) \)-generic (i.e. for any maximal antichain \( A \) of \( \mathbb{P}^M \) in \( M \), \( A \) is predense below \( q \)).
Definable forceings and strongly proper forceings

**Definition**

\(\mathcal{P}\): a partial order, \(n \geq 1\)

1. \(\mathcal{P}\) is *provably \(\Delta^1_n\)* if there are \(\Sigma^1_n\)-formula and \(\Pi^1_n\)-formula such that ZFC proves that they both define the pair \((\leq_{\mathcal{P}}, \perp_{\mathcal{P}})\).

2. \(\mathcal{P}\) is *strongly proper* if for any countable transitive model \(M\) with \((\mathcal{P}^M, \leq^M, \perp^M) \subseteq (\mathcal{P}, \leq, \perp)\) and for any \(p \in \mathcal{P}^M\), there is a \(q \leq p\) s.t. \(q\) is \((M, \mathcal{P})\)-generic (i.e. for any maximal antichain \(A\) of \(\mathcal{P}^M\) in \(M\), \(A\) is predense below \(q\)).

**Remark**

- Every provably \(\Delta^1_n\), strongly proper forcing is proper for each \(n\).
- Almost all known tree type forcings related to the reals are provably \(\Delta^1_2\) and strongly proper.
Arboreal forcings

Definition

\[ \mathbb{P}: \text{a partial order} \]

1. \( \mathbb{P} \) is *arboreal* if
   - conditions of \( \mathbb{P} \) are perfect trees on \( \omega \) (resp. on \( \{0, 1\} \)),
   - conditions are ordered by inclusion.

2. \( \mathbb{P} \) is *strongly arboreal* if
   - \( \mathbb{P} \) is arboreal,
   - for any \( T \in \mathbb{P} \) and \( t \in T \), \( T_t \) is in \( \mathbb{P} \),
     where \( T_t = \{ s \in T \mid \text{either } s \subseteq t \text{ or } t \subseteq s \} \).
Coding generic filters by reals

Remark

\( \mathbb{P} \): strongly arboreal, \( G \): \( \mathbb{P} \)-generic over \( V \). Put

\[
x_G = \bigcup \{ \text{stem}(T) \mid T \in G \}.
\]

Then \( x_G \) is an element of \( \omega^\omega \) (or of \( \{0,1\}^\omega \)) and \( V[x_G] = V[G] \). The real \( x_G \) is called a \( \mathbb{P} \)-generic real over \( V \).
Examples of strongly arboreal forcings

1. Cohen forcing ($\mathbb{C}$): for any $s$ in $\mathbb{C} = \omega^{<\omega}$, we assign

$$T_s = \{ t \in \omega^{<\omega} \mid t \supseteq s \}.$$ 

2. Random forcing ($\mathbb{B}$): conditions can be seen as perfect trees on $\{0, 1\}$ with positive Lebesgue measure.

3. Sacks forcing, Silver forcing, Mathias forcing, Hechler forcing, Laver forcing, Miller forcing etc.
Regularity properties for strongly arboreal forcings

Definition ($\mathbb{P}$-null sets, $I_\mathbb{P}$ and $I_\mathbb{P}^*$)

$\mathbb{P}$: strongly arboreal, $A$: a set of reals $A$ is $\mathbb{P}$-null if

$$(\forall T \in \mathbb{P}) \ (\exists T' \leq T) \ [T'] \cap A = \emptyset.$$ 

Let $I_\mathbb{P}$ be the $\sigma$-ideal generated by $\mathbb{P}$-null sets. $A$ is in $I_\mathbb{P}^*$ if $(\forall T \in \mathbb{P}) \ (\exists T' \leq T)$ such that $[T'] \cap A \in I_\mathbb{P}$.

Example

1. Cohen forcing
   - $\mathbb{C}$-null sets are nowhere dense sets.
   - $I_\mathbb{C} = I_\mathbb{C}^*$ is the meager ideal.

2. Random forcing
   - $\mathbb{B}$-null sets are Lebesgue null sets.
   - $I_\mathbb{B} = I_\mathbb{B}^*$ is the Lebesgue null ideal.
\( I_P \) vs \( I_P^* \)

**Question**

Let \( P \) be strongly arboreal, proper. Then \( I_P = I_P^* \)?

**Remark**

1. \( I_P \subseteq I_P^* \).
2. \( I_P = I_P^* \) if \( P \) is ccc or \( P \) admits a standard fusion argument.
Connection with “Forcing idealized”

Proposition

Let $\mathbb{P}$ be proper, strongly arboreal and define $i : \mathbb{P} \to \mathcal{B}/\mathbb{P}^*$ as follows:

\[ i(T) = \text{the equivalence class of } [T], \]

where $\mathcal{B}$ is the set of all Borel sets. Then $i$ is well-defined and dense.
Regularity properties for strongly arboreal forcings 2

Definition
\( \mathbb{P} \): strongly arboreal, \( A \): a set of reals.

\( A \) is **\( \mathbb{P} \)-measurable** if

\[
(\forall T \in \mathbb{P}) \ (\exists T' \leq T) \text{ either } [T'] \cap A \in I_\mathbb{P} \text{ or } [T'] \setminus A \in I_\mathbb{P}.
\]

Example

1. Cohen forcing: \( \mathbb{C} \)-measurability coincides with the Baire property.
2. Random forcing: \( \mathbb{B} \)-measurability coincides with Lebesgue measurability.
Regularity properties for strongly arboreal forcings 3

**Definition**

\( \mathbb{P} \): strongly arboreal, \( A \): a set of reals. 

\( A \) is \( \mathbb{P} \)-measurable if

\[
(\forall T \in \mathbb{P}) \ (\exists T' \leq T) \text{ either } [T'] \cap A \in l_\mathbb{P} \text{ or } [T'] \setminus A \in l_\mathbb{P}.
\]

**Example**

3 Sacks forcing: \( \mathcal{S} \)-measurability corresponds to not being a Bernstein set in the following sense:

If \( n \geq 1 \), \( \Gamma = \Sigma^1_n \) or \( \Pi^1_n \) or \( \Delta^1_n \), then the following are equivalent:

- every set in \( \Gamma \) is \( \mathcal{S} \)-measurable, and
- no sets in \( \Gamma \) are Bernstein.
Quasi-$P$-generic reals

**Definition**

$P$: strongly arboreal, $M$: a model, $x$: a real.

$x$ is *quasi-$P$-generic over* $M$ if for any Borel code $c$ in $M$, if $B_c$ is in $l_P^*$, then $x \notin B_c$.

**Example**

1. Cohen forcing: quasi-$\mathcal{C}$-generic reals over $M$ are the same as Cohen reals over $M$.

2. random forcing: quasi-$\mathcal{B}$-generic reals over $M$ are the same as random reals over $M$.

3. Sacks forcing: $x$ is quasi-$\mathcal{S}$-generic over $M$ iff $x$ is not in $M$. 
Quasi-generics vs generics

Remark

- $\mathbb{P}$-genericity implies quasi-$\mathbb{P}$-genericity if $\mathbb{P}$ is strongly proper, strongly arboreal and provably $\Delta^1_2$.
- Quasi-$\mathbb{P}$-genericity implies $\mathbb{P}$-genericity if $\mathbb{P}$ is additionally provably ccc.
Go back to Theorems...

**Theorem (1.)**

\[ \mathbb{P} : \text{proper, strongly arboreal.} \] Then the following are equivalent:

1. \( V \) is \( \Sigma^1_3 \)-\( \mathbb{P} \)-absolute,
2. every \( \Delta^1_2 \)-set of reals is \( \mathbb{P} \)-measurable.

**Theorem (1.)**

\[ \mathbb{P} : \text{proper, strongly arboreal, provably } \Delta^1_2. \] Assume

\[ \{ c \mid c: \text{a Borel code, } B_c \in I_{\mathbb{P}^*} \} \in \Sigma^1_2. \] \((*)\)

Then the following are equivalent:

1. \( V \) is \( \Sigma^1_3 \)-\( \mathbb{P} \)-absolute,
2. every \( \Delta^1_2 \)-set of reals is \( \mathbb{P} \)-measurable,
3. for any real \( a \) and any \( T \in \mathbb{P} \), there is a quasi-\( \mathbb{P} \)-generic real \( x \) in \([ T]\) over \( L[a] \).
On (*)

Question

If $\mathbb{P}$ is strongly proper, strongly arboreal and provably $\Delta^1_2$, then does the assumption (*) hold in the second theorem?

Remark

1. The set $\{c \mid c : \text{a Borel code, } B_c \in \mathbb{P}^*\}$ is $\Pi^1_2$ under the above assumptions for $\mathbb{P}$.
2. The condition (*) is true if $\mathbb{P}$ is provably ccc and $\Sigma^1_1$.

Conjecture (Zapletal)

For Mathias forcing, the condition (*) fails.

Remark

But the set of quasi-Mathias-generic reals over $L$ is $\Pi^1_2$!!
Some words for the proofs

**Theorem (1.)**

\( \mathbb{P} \): proper, strongly arboreal. Then the following are equivalent:

1. \( V \) is \( \Sigma^1_3 \)-\( \mathbb{P} \)-absolute,
2. every \( \Delta^1_2 \)-set of reals is \( \mathbb{P} \)-measurable.

**Proof.**

From (1) to (2):

- For a \( \Delta^1_2 \)-set \( A \) of reals, use Shoenfield trees and (1) to get the absolute tree representation of \( A \) between \( V \) and \( V^\mathbb{P} \).
- Use the fact that

\[ \{ x \mid x \text{ is a } \mathbb{P} \text{-generic real over } M \text{ for some suitable } M \} \]

is of measure one w.r.t. \( I^*_{\mathbb{P}} \).
Some words for the proofs

Theorem (1.)

$\mathbb{P}$: proper, strongly arboreal. Then the following are equivalent:

1. $V$ is $\Sigma^1_3$-$\mathbb{P}$-absolute,
2. every $\Delta^1_2$-set of reals is $\mathbb{P}$-measurable.

Proof.

From (2) to (1): Use the following to approximate (in $V$) the behavior of a witness (a $\mathbb{P}$-name for a real) of a $\Sigma^1_3$ fact in $V^\mathbb{P}$:

1. $\Sigma^1_2$-uniformization,
2. (2),
3. every $\mathbb{P}$-name for a real can be computed by a generic real in a Borel way (in $V$) (by properness).
Some words for the proofs 2

Theorem (1.)

\[ P: \text{strongly proper, strongly arboreal, provably } \Delta^1_2. \text{ Assume} \]

\[ \{ c \mid c: \text{a Borel code, } B_c \in l_{P^*} \} \in \Sigma^1_2. \tag{*} \]

Then the following are equivalent:

1. \( V \) is \( \Sigma^1_3 \)-\( P \)-absolute,
2. every \( \Delta^1_2 \)-set of reals is \( P \)-measurable,
3. for any real \( a \) and any \( T \in P \), there is a quasi-\( P \)-generic real \( x \) in \( [T] \) over \( L[a] \).

Proof.

From (2) to (1): the same as before.
From (1) to (3): Use (\( \ast \)) to show that “There is a quasi-\( P \)-generic real \( x \) in \( [T] \) over \( L[a] \)” is a \( \Sigma^1_3 \) statement.
Some words for the proofs 2

Proof.

From (3) to (2):

1. Case 1: $\omega_1^V > \omega_1^{L[a]}$ for any real $a$.
   - Capture a given $\Delta^1_2$-set of reals in $L[a]$ for some real $a$ by using Shoenfield trees.
   - Use the fact that
     \[
     \{ x \mid x \text{ is a } P^{L[a]}\text{-generic real over } L[a] \}
     \]
     is of measure one w.r.t. $I_P^*$.

2. Case 2: $\omega_1^V = \omega_1^{L[a]}$ for some real $a$.
   Get the absolute decomposition of $\Sigma^1_2$-sets into Borel sets between $L[a]$ and $V$ by using the absoluteness of Shoenfield trees between them, and use (3).
In ZFC, the equivalence fails in general (i.e. $\Sigma^1_4$-forcing absoluteness does not imply $\Delta^1_3$-regularity property and vice versa). We need a natural additional assumption.
\[ \Sigma^1_4 \text{-} \text{forcing absoluteness and } \Delta^1_3 \text{-regularity properties} \]

**Theorem**

\( P \): proper, strongly arboreal

Assume every set has a sharp. Then either \( \Delta^1_2 \)-determinacy holds or the following are equivalent:

1. \( V \) is \( \Sigma^1_4 \)-\( P \)-absolute,
2. every \( \Delta^1_3 \)-set of reals is \( P \)-measurable.
\( \Sigma^1_4 \)-forcing absoluteness and \( \Delta^1_3 \)-regularity properties

**Theorem**

\( \mathbb{P} \): proper, strongly arboreal

Assume every set has a sharp. Then either \( \Delta^1_2 \)-determinacy holds or the following are equivalent:

1. \( V \) is \( \Sigma^1_4 \)-\( \mathbb{P} \)-absolute,
2. every \( \Delta^1_3 \)-set of reals is \( \mathbb{P} \)-measurable.

**Question**

Does \( \Delta^1_2 \)-determinacy imply that \( V \) is \( \Sigma^1_4 \)-\( \mathbb{P} \)-absolute for any proper, strongly arboreal \( \mathbb{P} \)?

**Remark**

1. If every set has a sharp, (1) implies (2).
2. The answer of the question is ‘Yes’ if \( \mathbb{P} \) is ccc.
Theorem (1.)

\( \mathbb{P} \): strongly proper, strongly arboreal, provably \( \Delta^1_2 \).
Assume every set has a sharp.
Then either \( \Delta^1_2 \)-determinacy holds or the following are equivalent:

1. \( \mathbb{V} \) is \( \Sigma^1_4 \)-\( \mathbb{P} \)-absolute,
2. every \( \Delta^1_3 \)-set of reals is \( \mathbb{P} \)-measurable,
3. for any real \( a \) and any \( T \in \mathbb{P} \), there is a quasi-\( \mathbb{P} \)-generic real \( x \) in \( [T] \) over \( K_a \), where

\[
K_a = \begin{cases} 
  \text{the Mitchell-Steel core model} & \text{if } a^+ \text{ exists,} \\
  \text{the Dodd-Jensen core model} & \text{otherwise.}
\end{cases}
\]
Comparison between previous case and present case

<table>
<thead>
<tr>
<th>Previous</th>
<th>Present 1</th>
<th>Present 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>$K$</td>
<td>$\Delta^1_2$-determinacy</td>
</tr>
<tr>
<td>$\Sigma^1_2$-correctness</td>
<td>$\Sigma^1_3$-correctness</td>
<td>irrelevant</td>
</tr>
<tr>
<td>Shoenfield trees ($\omega_1$)</td>
<td>Martin-Solovay trees ($\nu_2$)</td>
<td>irrelevant</td>
</tr>
<tr>
<td>$\Sigma^1_2$-uniformization</td>
<td>$\Sigma^1_3$-uniformization</td>
<td>$\Pi^1_3$-uniformization</td>
</tr>
</tbody>
</table>
Comparison between previous case and present case

<table>
<thead>
<tr>
<th>Previous</th>
<th>Present 1</th>
<th>Present 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>$K$</td>
<td>$\Delta^1_2$-determinacy</td>
</tr>
<tr>
<td>$\Sigma^1_2$-correctness</td>
<td>$\Sigma^1_3$-correctness</td>
<td>irrelevant</td>
</tr>
<tr>
<td>Shoenfield trees ($\omega_1$)</td>
<td>Martin-Solovay trees ($u_2$)</td>
<td>irrelevant</td>
</tr>
<tr>
<td>$\Sigma^1_2$-uniformization</td>
<td>$\Sigma^1_3$-uniformization</td>
<td>$\Pi^1_3$-uniformization</td>
</tr>
</tbody>
</table>

**Question (Sharps for sets vs sharps for reals)**

Assume every real has a sharp and let $\mathbb{P}$ be proper and provably $\Delta^1_2$. Then does every real have a sharp in $V^{\mathbb{P}}$ and $u^V_2 = u^{V^{\mathbb{P}}}_2$?
Thank you!