Representations of quantized function algebras and the transition matrices from Canonical bases to PBW bases

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Quantized enveloping algebras

- \( \mathfrak{g} \) a finite dimensional complex simple Lie algebra
- \( U_q(\mathfrak{g}) = \langle E_i, F_i, K_i | i \in I \rangle_{Q(q)} \)-algebra
  the quantized enveloping algebra /\( Q(q) \) (a \( q \)-analogue of \( U(\mathfrak{g}) \))
- \( U_q(\mathfrak{n}^+) = \langle E_i | i \in I \rangle_{Q(q)} \)-algebra

The quantized enveloping algebra \( U_q(\mathfrak{g}) \) has a Hopf algebra structure. In particular, its coproduct \( \Delta \) is defined as follows:

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{-i} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i.
\]
Some operations

Definition

We define the $\mathbb{Q}(q)$-algebra involution $\omega : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ by

$$\omega(E_i) = F_i, \ \omega(F_i) = E_i, \ \omega(K_i) = K_i^{-1}.$$

We define the $\mathbb{Q}(q)$-algebra anti-involution $\ast : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ by

$$\ast(E_i) = E_i, \ \ast(F_i) = F_i, \ \ast(K_i) = K_i^{-1}.$$

We define the $\mathbb{Q}$-algebra involution $\overline{\cdot} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ by

$$\overline{E_i} = E_i, \ \overline{F_i} = F_i, \ \overline{K_i} = K_i^{-1}, \ \overline{q} = q^{-1}.$$
Let $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ be a reduced word of the longest element $w_0$ of the Weyl group $W$. (i.e. $w_0 = s_{i_1}s_{i_2} \cdots s_{i_N}$. In particular, $N := \text{the length of } w_0$.)

**Definition (The PBW bases)**

The vectors

$$\begin{align*}
\{ E_i^c & : E_{i_1}^{(c_1)} T'_{i_1,1} (E_{i_2}^{(c_2)}) \cdots T'_{i_1,1} T'_{i_2,1} \cdots T'_{i_{N-1},1} (E_{i_N}^{(c_N)}) \}_c
\end{align*}$$

$(c = (c_1, c_2, \ldots, c_N) \in (\mathbb{Z}_{\geq 0})^N)$ forms a basis of $U_q(n^+)$. Here, $T'_{i,1}$ is a $q$-analogue of “the action of the braid group”.

**Remark**

For any reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ of $w_0$, we have

$$\Delta_+ = \{ \beta_{i_1}^1, \beta_{i_2}^2, \ldots, \beta_{i_N}^N \} \text{ where } \beta_{i}^k := s_{i_1} \cdots s_{i_{k-1}} (\alpha_{i_k}).$$
Let \( i \) be a reduced word of \( w_0 \). Then, there uniquely exists a basis \( \{ G^{(c)} \}_c \) of \( U_q(n^+) \) such that

- \( G^{(c)} = G^{(c)} \),
- \( G^{(c)} = E_i^c + \sum_{d > c} \zeta_d^c E_i^d \) with \( \zeta_d^c \in q\mathbb{Z}[q] \).

We consider the lexicographic order on \((\mathbb{Z}_{\geq 0})^N\).

**Definition (The canonical basis)**

We call \( \{ G^{(c)} \}_c \) the canonical basis of \( U_q(n^+) \).

**Remark**

The definition of canonical basis does not depend on the choice of \( i \). (The data \((c)\) depend on \( i \).)
Our main theorem

**Theorem (Positivity)**

Assume that the Lie algebra \( \mathfrak{g} \) is of type ADE. Take an arbitrary reduced word \( i \) of \( w_0 \). Then, for any \( c \in (\mathbb{Z}_{\geq 0})^N \), we have

\[
G^{(c)} = E_i^c + \sum_{d > c} i \zeta_d^c E_i^d \quad \text{with} \quad i \zeta_d^c \in q \mathbb{N}[q].
\]

**Remark**

In general, it is difficult to describe the explicit form of the element of the canonical basis.
The positivity of these coefficients was originally proved by

- Lusztig (1990) : for the “adapted” reduced word $i$ of $w_0$ (via “geometric realization”)
- Kato (2014) : for the arbitrary case (via “categorification”)

We gave a new algebraic proof of the above theorem from now on. (It has been recently found that our “calculation procedure” is same as a certain other calculation procedure.)

From now on, we again assume that $g$ is a finite dimensional complex simple Lie algebra.
Quantized function algebras

The dual space $U_q(\mathfrak{g})^*$ of $U_q(\mathfrak{g})$ has a $\mathbb{Q}(q)$-algebra structure induced from the coalgebra structure of $U_q(\mathfrak{g})$.

Definition (The quantized function algebra)

The quantized function algebra $\mathbb{Q}_q[G]$ is a subalgebra of $U_q(\mathfrak{g})^*$ generated (in fact, spanned) by the matrix coefficients

$$c_{f,v}^\lambda \mapsto (u \mapsto \langle f, u.v \rangle),$$

here,

- $\lambda \in P_+ (= \text{the set of dominant integral weight})$,
- $V(\lambda)$ the integrable highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda$,
- $f \in V(\lambda)^*, v \in V(\lambda)$.

Then, $\mathbb{Q}_q[G]$ has a Hopf algebra structure induced from the one of $U_q(\mathfrak{g})$ and a left and right $U_q(\mathfrak{g})$-algebra structure.
\( \mathbb{Q}_q[G] \) - a quantum analogue of the algebra of regular functions on \( G \) (\( G \) is the connected simply connected simple complex algebraic group whose Lie algebra is \( \mathfrak{g} \).)

The algebra \( \mathbb{Q}_q[G] \) has infinite dimensional irreducible modules [This point is extremely different from the classical(=“q = 1”) situation!!]:

\[
\mathbb{Q}_q[G] \to \mathbb{Q}_q[SL_2] \curlysim V_i := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q)|m\rangle_i.
\]

(dual to \( U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{g}) \).)
Theorem (Soibelman (1990))

Let $w \in W$. Then, for any reduced expression $w = s_{i_1} \cdots s_{i_l}$, the $\mathbb{Q}_q[G]$-module $V_{i_1} \otimes \cdots \otimes V_{i_l}$ is irreducible and its isomorphism class does not depend on the choice of the reduced expressions. Hence, we denote this module by $V_w$. 
Strategy 1

**Theorem (O-)**

For $x \in U_q(n^+), \lambda \in P_+$ and a reduced word $i$ of $w_0$, we write

$$x = \sum_{c \in (\mathbb{Z}_{\geq 0})^N} i \zeta^x_c E_i^c \quad \text{with} \quad i \zeta^x_c \in \mathbb{Q}(q), \quad \text{and} \quad i \zeta^x_c$$

$$(c^\lambda_{f_{\lambda},v_{w_0 \lambda} \cdot \ast (x)} \cdot |(0)\rangle \rangle_i = \sum_{c \in (\mathbb{Z}_{\geq 0})^N} i \zeta^\lambda_{c,x} |(c)\rangle \rangle_i \quad \text{with} \quad i \zeta^\lambda_{c,x} \in \mathbb{Q}(q)(\text{in } V_{w_0}).$$

(Here, $f_{\lambda}$ is a highest weight vector of $V(\lambda)^*$ which sends a fixed highest weight vector $v_\lambda$ of $V(\lambda)$ to 1, and $v_{w_0 \lambda}$ is the lowest weight lower global basis element of $V(\lambda).$)

When $\lambda \in P_+$ tends to $\infty$ in the sense that $\langle \lambda, \alpha^Y_i \rangle$ tends to $\infty$ for all $i \in I, i \zeta^\lambda_{c,x}$ converges to $i \zeta^x_c$ in the complete discrete valuation field $\mathbb{Q}((q)).$
Strategy 2

For sufficiently large $L$, we set $\lambda_0 := 2(N + 1) L \rho$. $\rho :=$ the Weyl vector. Then, by the calculation method of the previous theorem, we can obtain

$$
\left( e^{\lambda_0 \lambda_0} \cdot \ast (G^{(c)}) \right) \cdot |(0)\rangle \rangle_i = \sum_{d \geq c} i \xi_d^c |(d)\rangle \rangle_i + q^L \sum_{d' \in (\mathbb{Z}_{\geq 0})^N} \eta_{d'} |(d')\rangle \rangle_i
$$

with $\eta_{d'} \in \mathbb{Z}[q]$.

$$
(=: \sum_{d \in (\mathbb{Z}_{\geq 0})^N} i \xi_{d'}^c |(d)\rangle \rangle_i)
$$
On the other hand, we calculate the left-hand side of the previous equality as follows:

\[
\langle \sum_{b_1', \ldots, b_{N-1}' \in B(\lambda_0)} c^{\lambda_0}_{(G^{\text{low}}_{\lambda_0}(b_0'), \cdot), G^{\text{up}}_{\lambda_0}(b_1') \cdot |0\rangle i_1 \otimes c^{\lambda_0}_{(G^{\text{low}}_{\lambda_0}(b_1'), \cdot), G^{\text{up}}_{\lambda_0}(b_2') \cdot |0\rangle i_2}
\]

Here,

- \((\ ,\ ) : V(\lambda_0) \times V(\lambda_0) \to \mathbb{Q}(q)\) the “good” \(\mathbb{Q}(q)\)-bilinear form
- \(\{ G^{\text{low/ up}}_{\lambda_0}(b') \}_{b' \in B(\lambda_0)}\) the lower/upper global basis of \(V(\lambda_0)\)
- \(G^{\text{low}}_{\lambda_0}(b_0') := \omega(G^{(c)})(.v_{\lambda_0})\)
We can deduce that:

**Proposition**

For each \( k \),
\[
\langle \lambda_0 \rangle_{\chi^\text{low}_0(b'_{k-1})}, G^\text{up}_0(b'_k) \cdot |0\rangle \rangle_{i_k} = p_k |c\rangle \rangle_{i_k},
\]

with \( c := -\frac{1}{2} \langle \text{wt } b'_{k-1} + \text{wt } b'_k, \alpha_{i_k}^\vee \rangle \) and \( p_k \in q^{-L} \mathbb{Z}[q] \).

- \( i \zeta^c \in \mathbb{Z}[q^{\pm 1}] \), and
- we may ignore the degree \( \geq NL \) part of the Laurent polynomial \( p_k \) for any \( k \) when calculating the degree \( < L \) parts of the Laurent polynomials \( i \zeta^c \).

Key: “the positivity of \( q \)-derivations” (Lusztig)
Remark

In our calculation, we use the following property of canonical bases:

Proposition (Similarity of the structure constants)

We set

\[ F_i^{(p)} G_{\text{low}}^\text{low}(b) = \sum_{\tilde{b} \in B(\infty)} \tilde{c}^\tilde{b}_{-p, i, b} G_{\text{low}}^\text{low}(\tilde{b}), \]

\[ (e'_i)^{p}(G_{\text{low}}^\text{low}(b)) = \sum_{\tilde{b} \in B(\infty)} \tilde{d}^{i, p}_{b, i, \tilde{b}} G_{\text{low}}^\text{low}(\tilde{b}). \]

Then, for any \( b, \hat{b} \in B(\infty), \; i \in I \) and \( p \in \mathbb{Z}_{\geq 0} \), we have

\[ \left( c^\hat{b}_{-p, i, \hat{b}} \right) < -\Delta_i(d-1)p = \left( q_i^{\frac{1}{2}d(d-1)} \begin{bmatrix} \varepsilon_i(\hat{b}) \\ p \end{bmatrix} \tilde{d}^{i, d}_{b, \varepsilon_i(\hat{b}), \hat{b}} \right) < -\Delta_i(d-1)p, \]

where \( d := \varepsilon_i(\hat{b}) - p \).

Reference: arXiv1501.01416 (Slides: http://www.ms.u-tokyo.ac.jp/~oya)