Quantum Grothendieck ring isomorphisms for quantum affine algebras of type A and B

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Motivation (1)

Topic: Finite dimensional representations of affine quantum groups

Question 1

Dimensions/$q$-characters of simple modules?

- ∃ Classification of simple modules [Chari-Pressley 1990’s]
  “Highest weight theory”
- However, there are NO known closed formulae of their dimensions and $q$-characters in general. (e.g. ∄ analogue of Weyl-Kac character formulae...)

Question 2

Description of representation rings and their “deformations”? 

- Some (deformed) representation rings are known to be described nicely as (quantum) cluster algebras...
Question 1

Dimensions/$q$-characters of simple modules?

- **ADE case** $\exists$ algorithm to compute them! [Nakajima ’04]
  “Kazhdan-Lusztig algorithm”
  The tool is $t$-deformed $q$-character, and the geometric construction (via quiver varieties) of simple modules guarantees this algorithm.

- **Arbitrary (untwisted) case** [Hernandez ’04]
  - $\exists$ $t$-deformed $q$-characters, defined algebraically
    ( $\not\exists$ geometry for non-symmetric cases)
  - Kazhdan-Lusztig algorithm gives conjectural $q$-characters of simple modules

However, they are still candidates in non-symmetric cases.
Question 2

Description of representation rings and their “deformations”? 

- [Hernandez-Leclerc ’10 –, Kang-Kashiwara-Kim-Oh ’15, Oh-Suh ’16] The category of finite dimensional modules of affine quantum groups has several interesting monoidal subcategories ($C_{\mathbb{Z}}, C_{\mathbb{Z}}, C_{\ell}, \ell \in \mathbb{Z}, C_{\mathbb{Q}}$ etc.), which are expected to be “monoidal categorifications” of cluster algebras (this fact is indeed proved in many cases).
Question 2

*Description of representation rings and their “deformations”?*

- **$X = \text{ADE}$** case
  - \( K_t(\mathcal{C}_{Q,X_n^{(1)}}) \) the \( t \)-deformed Grothendieck ring (=quantum Grothendieck ring) of \( \mathcal{C}_{Q,X_n^{(1)}} \) for type \( X_n^{(1)} \)
  - \( \mathcal{A}_v[\mathbb{N}_X^{X_n}] \) the quantized coordinate algebra of the unipotent group of type \( X_n \) (\( \exists \) quantum cluster algebra structure !)
  
  (Each terminology will be explained later.)

**Theorem (Hernandez-Leclerc ’15)**

\[
K_t(\mathcal{C}_{Q,X_n^{(1)}}) \cong \mathcal{A}_v[\mathbb{N}_X^{X_n}], \quad \left\{ \begin{array}{l}
(q,t)\text{-characters of simple modules} \\
\end{array} \right\} \leftrightarrow \text{dual canonical basis.}
\]

Does it also hold in non-symmetric cases?

Hironori OYA (IMJ-PRG)
Quantum Grothendieck ring isomorphisms
June 14, 2018
Overview of Main results

In this talk, we consider the case of type $B_n^{(1)}$. Let $C_{Q,B_n^{(1)}}$ be the monoidal subcategory $C_Q$ for type $B_n^{(1)}$.

**Theorem (Hernandez-O.)**

\[
K_t(C_{Q,B_n^{(1)}}) \cup \{ \text{(q, t)-characters of simple modules} \} \cong A_v[N_{A_{2n-1}}] \cup \{ \text{dual canonical basis} \} \cong K_t(C_{Q',A_{2n-1}^{(1)}}) \cup \{ \text{(q, t)-characters of simple modules} \}
\]

**Remark**

There are no known direct relations between the quantum affine algebras of type $B_n^{(1)}$ and $A_{2n-1}^{(1)}$ themselves!
Overview of Main results

Let $\mathcal{C}_{Q,B_n^{(1)}}$ be the monoidal subcategory $\mathcal{C}_Q$ for type $B_n^{(1)}$.

**Theorem (Hernandez-O.)**

\[
K_t(\mathcal{C}_{Q,B_n^{(1)}}) \cup \left\{ (q,t)-\text{characters of simple modules} \right\} \cong A_v[N_{A_{2n-1}}^A] \cup \text{dual canonical basis} \cong [\text{HL}] K_t(\mathcal{C}_{Q',A_{2n-1}^{(1)}}) \cup \left\{ (q,t)-\text{characters of simple modules} \right\}
\]

Kashiwara-Oh established an isomorphism between $K_{t=1}^{(1)}(\mathcal{C}_{Q,B_n^{(1)}})$ and $\mathbb{C}[N_{A_{2n-1}}^A]$ by a different method. Combining this result with our theorem above, we obtain the following:

**Theorem (Hernandez-O.)**

The $(q,t)$-characters of simple modules in $\mathcal{C}_{Q,B_n^{(1)}}$ specialize to the corresponding $q$-characters.
Let $\mathcal{C}_{Q,B_n^{(1)}}$ be the monoidal subcategory $\mathcal{C}_Q$ for type $B_n^{(1)}$.

**Theorem (Hernandez-O.)**

$$K_t(\mathcal{C}_{Q,B_n^{(1)}}) \cup \{ (q,t)\text{-characters of simple modules} \} \cong A_v[N_{A_{2n-1}}] \cup \{ \text{dual canonical basis} \} \cong K_t(\mathcal{C}_{Q',A_{2n-1}^{(1)}}) \cup \{ (q,t)\text{-characters of simple modules} \}$$

**Theorem (Hernandez-O.)**

*The $(q,t)$-characters of simple modules in $\mathcal{C}_{Q,B_n^{(1)}}$ specialize to the corresponding $q$-characters.*

$\leadsto$ The Kazhdan-Lusztig algorithm gives “correct” answers in $\mathcal{C}_{Q,B_n^{(1)}}$!
Quantum affine algebras

Let

- \( g \) a finite dimensional simple Lie algebra over \( \mathbb{C} \)
- \( \mathcal{L}g := g \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}] \) its loop algebra, \([X \otimes t^m, Y \otimes t^{m'}] = [X, Y] \otimes t^{m+m'}\)
- \( \mathcal{U}_q(\mathcal{L}g) \) the Drinfeld-Jimbo quantum loop algebra over \( \mathbb{C} \) with a parameter \( q \in \mathbb{C}^\times \) not a root of unity

Generators:
\[
\{ k_i^{\pm 1}, x_i^\pm, h_i, s | i \in I, r \in \mathbb{Z}, s \in \mathbb{Z} \setminus \{0\} \}
\]

Properties

- \( \mathcal{U}_q(\mathcal{L}g) \) has a Hopf algebra structure.
- \( \mathcal{U}_q(g) \hookrightarrow \mathcal{U}_q(\mathcal{L}g) \), \( e_i \mapsto x_{i,0}^+, f_i \mapsto x_{i,0}^-, k_i^{\pm 1} \mapsto k_i^{\pm 1} \).

Let \( \mathcal{C} \) be the category of finite-dimensional \( \mathcal{U}_q(\mathcal{L}g) \)-modules of type 1 (i.e. the eigenvalues of the actions of \( \{ k_i | i \in I \} \) are of the form \( q^m, m \in \mathbb{Z} \)).

Remark: \( \mathcal{C} \) is a non-semisimple abelian \( \otimes \)-category.
Let $V \in \mathcal{C}$. Frenkel-Reshetikhin showed that

$$\left\{ \text{Generalized simultaneous eigenvalues of all } k_i^{\pm 1}, h_i, s \right\} \overset{1:1}{\leftrightarrow} \left\{ \text{Laurent monomials } m \text{ in } Y_{i,a}'s \ (i \in I, a \in \mathbb{C}^\times) \right\} \rightleftharpoons V = \bigoplus_m V_m,$$ called the $\ell$-weight space decomposition.

$Y_{i,a}$ is an “affine analogue” of $e^{-\varpi_i}$, $\varpi_i$ fundamental weight.

Define the $q$-character of $V$ as

$$\chi_q(V) := \sum_m \dim(V_m)m.$$

Then $\chi_q$ defines an injective algebra homomorphism

$$\chi_q : K(C) \rightarrow \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times] =: \mathcal{Y}_{\mathbb{C}^\times},$$

here $K(C)$ be the Grothendieck ring of $\mathcal{C}$ [Frenkel-Reshetikhin].

$K(C)$ is commutative. (However sometimes $V \otimes W \not\cong W \otimes V$ in $\mathcal{C}$.)
Set $\mathcal{B}_{\mathbb{C}^\times} := \left\{ \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{m_{i,a}} \mid m_{i,a} \geq 0 \right\} \subset \mathcal{Y}_{\mathbb{C}^\times}$ dominant monomials.

**Classification of simple modules [Chari-Pressley]**

There is a one-to-one correspondence:

$\{\text{simple modules in } \mathcal{C}\} / \sim \leftrightarrow \mathcal{B}_{\mathbb{C}^\times}$

$\mathcal{B}_{\mathbb{C}^\times} \cup [L(m)] \leftrightarrow m$

$\exists$ an “affine analogue” $A_{i,a} \in \mathcal{Y}_{\mathbb{C}^\times}$ of $e^{\alpha_i}$, $\alpha_i$ simple root.

**Type $A_n^{(1)}$**

$A_{i,a} = Y_{i,a}q^{-1}Y_{i,a}^{-1}Y_{i-1,a}^{-1}Y_{i+1,a}^{-1} \quad (\longleftrightarrow e^{\alpha_i} = e^{2\varpi_i - \varpi_{i-1} - \varpi_{i+1}})$

$(Y_{0,a} = Y_{n+1,a} := 1, \ e^{\varpi_0} = e^{\varpi_{n+1}} := 1)$
**q-characters (2)**

Set \( B_{\mathcal{C}^\times} := \left\{ \prod_{i \in I, a \in \mathcal{C}} Y_{i,a}^{m_{i,a}} \mid m_{i,a} \geq 0 \right\} \subset \mathcal{V}_{\mathcal{C}^\times} \) dominant monomials.

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**Classification of simple modules [Chari-Pressley]**

There is a one-to-one correspondence:

\[
\left\{ \text{simple modules in } \mathcal{C} \right\} / \sim \leftrightarrow B_{\mathcal{C}^\times}
\]

\[
\bigcup \left[ L(m) \right] \leftrightarrow m
\]

\( \exists \) an “affine analogue” \( A_{i,a} \in \mathcal{V}_{\mathcal{C}^\times} \) of \( e^{\alpha_i} \), \( \alpha_i \) simple root.

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**Type \( B_{(1)}^n \)**

\[
A_{i,a} = \begin{cases} 
Y_{i,a}q^{-2}Y_{i,a}q^2Y_{i-1,a}^{-1}Y_{i+1,a}^{-1} & \text{if } i \leq n - 2 \\
Y_{n-1,a}q^{-2}Y_{n-1,a}q^2Y_{n-2,a}^{-1}Y_{n,a}^{-1}Y_{j,a}^{-1} & \text{if } i = n - 1 \\
Y_{n-1,a}q^{-1}Y_{n,a}Y_{n-1,a}^{-1} & \text{if } i = n.
\end{cases}
\]

\( (Y_{0,a} := 1) \)
Set $\mathcal{B}_{\mathbb{C}^\times} := \left\{ \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{m_{i,a}} \mid m_{i,a} \geq 0 \right\} \subset \mathcal{Y}_{\mathbb{C}^\times}$ dominant monomials.

**Classification of simple modules [Chari-Pressley]**

There is a one-to-one correspondence:

$$\{\text{simple modules in } \mathcal{C}\} / \sim \leftrightarrow \mathcal{B}_{\mathbb{C}^\times} \cup \bigcup [L(m)] \leftrightarrow m$$

$\exists$ an “affine analogue” $A_{i,a} \in \mathcal{Y}_{\mathbb{C}^\times}$ of $e^{\alpha_i}$, $\alpha_i$ simple root.

Define the partial ordering on the set of Laurent monomials in $\mathcal{Y}_{\mathbb{C}^\times}$ as

$$m \geq m' \iff m^{-1}m' \text{ is a product of } A_{i,a}^{-1} \text{'s.}$$

**Theorem (Frenkel-Mukhin)**

$$\chi_q(L(m)) = m + \text{(sum of terms lower than } m), \forall m \in \mathcal{B}_{\mathbb{C}^\times}.$$
$q$-characters (3)

$\mathcal{C}_\bullet :=$ the full subcategory of $\mathcal{C}$ such that 

\[
\text{object : } V \text{ with } \chi_q(V) \in \mathbb{Z}[Y_{i,q}^{\pm 1} \mid i \in I, r \in \mathbb{Z}] =: \mathcal{Y}.
\]

Properties

- $\mathcal{C}_\bullet$ is a (non-semisimple) abelian $\otimes$-subcategory.
- $\mathcal{C} = \bigotimes_{a \in \mathbb{C}^\times / q\mathbb{Z}} (\mathcal{C}_\bullet)_a$ ($(\mathcal{C}_\bullet)_a$ is obtained from $\mathcal{C}_\bullet$ by shift of the spectral parameter by $a$).

From now on, we always work in $\mathcal{C}_\bullet$, and write

\[
Y_{i,r} := Y_{i,q^r} \quad A_{i,r} := A_{i,q^r} \quad \mathcal{B} := \mathcal{B}_{\mathbb{C}^\times} \cap \mathcal{Y}.
\]

Example

- $\mathfrak{g} = \mathfrak{sl}_2, I = \{1\}$, $\chi_q(L(Y_{1,r})) = Y_{1,r} + Y_{1,r+2}^{-1} = Y_{1,r}(1 + A_{1,r+1}^{-1})$.
- $\mathfrak{g} = \mathfrak{so}_5, I = \{1, 2\}$,

\[
\chi_q(L(Y_{1,r})) = Y_{1,r} + Y_{2,r+1}Y_{2,r+3}Y_{1,r+4}^{-1} + Y_{2,r+1}Y_{2,r+5}^{-1} + Y_{1,r+2}Y_{2,r+3}^{-1}Y_{2,r+5}^{-1} + Y_{1,r+6}^{-1}.
\]
$C_\bullet$ := the full subcategory of $C$ such that

object : $V$ with $\chi_q(V) \in \mathbb{Z}[Y_{i,q}^{\pm 1} \mid i \in I, r \in \mathbb{Z}] =: \mathcal{Y}$.

**Properties**

- $C_\bullet$ is a (non-semisimple) abelian $\otimes$-subcategory.
- $C = \bigotimes_{a \in \mathbb{C}^\times / q\mathbb{Z}} (C_\bullet)_a$ (($C_\bullet)_a$ is obtained from $C_\bullet$ by shift of the spectral parameter by $a$).

From now on, we always work in $C_\bullet$, and write

$$Y_{i,r} := Y_{i,q^r}, \quad A_{i,r} := A_{i,q^r}, \quad \mathcal{B} := \mathcal{B}_{\mathbb{C}^\times} \cap \mathcal{Y}.$$ For $m = \prod_{i \in I, r \in \mathbb{Z}} Y_{i,r}^{u_{i,r}} \in \mathcal{B}$, a standard module is defined as

$$M(m) := \bigotimes_{r \in \mathbb{Z}} \left( \bigotimes_{i \in I} L(Y_{i,r})^{\otimes u_{i,r}} \right).$$

$\sim \{ [L(m)] \mid m \in \mathcal{B} \}$ and $\{ [M(m)] \mid m \in \mathcal{B} \}$ are $\mathbb{Z}$-bases of $K(C_\bullet)$. 
Quantum Grothendieck rings (1)

We follow Hernandez's algebraic construction of quantum Grothendieck rings here.

**Remark**

∃ other (geometric) constructions given by Varagnolo-Vasserot or Nakajima for ADE cases, and all constructions produce equivalent rings in these cases.

First, we prepare a deformation $\mathcal{Y}_t$ of the ambient Laurent polynomial ring $\mathcal{Y}$.

$\mathcal{Y}_t$ is a $\mathbb{Z}[t^{\pm 1/2}]$-algebra such that

- **generators**: $\tilde{Y}_{i,r}$ ($i \in I$, $r \in \mathbb{Z}$) and their inverses $\tilde{Y}_{i,r}^{-1}$
- **relations**: $\tilde{Y}_{i,r}$'s mutually $t$-commute.

E.g. $B_2^{(1)}$-case: $\tilde{Y}_{1,r+2} \tilde{Y}_{1,r} = t \tilde{Y}_{1,r} \tilde{Y}_{1,r+2}$, $\tilde{Y}_{1,r+5} \tilde{Y}_{2,r} = t^{-1} \tilde{Y}_{2,r} \tilde{Y}_{1,r+5}$, ...
There exists a \( \mathbb{Z} \)-algebra homomorphism \( \text{ev}_{t=1} : \mathcal{Y}_t \to \mathcal{Y} \) given by

\[
t^{1/2} \mapsto 1 \quad \quad \tilde{Y}_{i,r} \mapsto Y_{i,r}.
\]

This map is called the specialization at \( t = 1 \).

There exists a \( \mathbb{Z} \)-algebra anti-involution \( (\cdot) \) on \( \mathcal{Y}_t \) given by

\[
t^{1/2} \mapsto t^{-1/2} \quad \quad \tilde{Y}_{i,r} \mapsto t^{-1} \tilde{Y}_{i,r}.
\]

This map is called the bar-involution.

\[\forall \ m \in \mathcal{Y} \text{ monomial} \mapsto \exists! \ \overline{m} \in \mathcal{Y}_t \text{ monomial (with coefficient in } t^{\mathbb{Z}}) \text{ such that } \overline{\overline{m}} = m. \quad (\text{e.g. } \overline{Y}_{i,r} = t^{-1/2} \tilde{Y}_{i,r}. ) \quad \text{Set } \overline{A}_{i,r} := \overline{A}_{i,r}.\]
For $i \in I$, set

$$K_{i,t} := \langle \tilde{Y}_{i,r}(1 + t\tilde{A}_{i,r+r_i}^{-1}), \tilde{Y}_{j,r} \mid j \in I \setminus \{i\}, r \in \mathbb{Z} \rangle_{\mathbb{Z}[t^{\pm 1/2}]_{-\text{alg.}}} \subset \mathcal{Y}_t.$$ 

Define the quantum Grothendieck ring of $C_\bullet$ as

$$K_t(C_\bullet) := \bigcap_{i \in I} K_{i,t}.$$ 

**Remark**

Indeed, $K_{i,t}$ is the kernel of a $t$-analogue of “the screening operator associated to $i \in I$” [Hernandez].

$\implies K_t(C_\bullet)$ is an affine analogue of the space of “$W$-invariant functions”.

**Theorem (Varagnolo-Vasserot, Nakajima, Hernandez)**

$$\text{ev}_{t=1}(K_t(C_\bullet)) = K(C_\bullet).$$
\( (q, t) \)-characters (1)

\[ \exists \ a \ \mathbb{Z}[t^{\pm 1/2}] \text{-basis} \ \{M_t(m) \mid m \in B\} \text{ of } K_t(C_{\bullet}) \text{ such that} \]
\[ \text{ev}_{t=1}(M_t(m)) = \chi_q(M(m)) \] [Nakajima, Hernandez].
\[ \rightsquigarrow \ M_t(m) \text{ is called the } (q, t)\text{-character of } M(m). \]

All \( M_t(m) \) can be explicitly calculated once we know \( M_t(Y_{i,0}), i \in I \).

Theorem (Nakajima (ADE cases), Hernandez (arbitrary))

\[ \exists \! \{L_t(m) \mid m \in B\} \ a \ \mathbb{Z}[t^{\pm 1/2}] \text{-basis of } K_t(C_{\bullet}) \text{ such that} \]
(S1) \[ L_t(m) = L_t(m), \text{ and} \]
(S2) \[ M_t(m) = L_t(m) + \sum_{m' < m} P_{m,m'}(t)L_t(m') \text{ with} \]
\[ P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]. \]

The element \( L_t(m) \) is called the \( (q, t)\text{-character of } L(m). \)
$(q,t)$-characters (2)

(S1) $L_t(m) = L_t(m)$ (S2) $M_t(m) = L_t(m) + \sum_{m' < m} P_{m,m'}(t)L_t(m')$, $P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]

**Remark**

The characterization properties (S1) and (S2) provide an inductive algorithm for computing $P_{m,m'}(t)$'s, called Kazhdan-Lusztig algorithm.

When $g$ is of ADE type,

$$ev_{t=1}(L_t(m)) = \chi_q(L(m)) \text{ [Nakajima]}.$$ 

Its proof is based on his geometric construction using quiver varieties, and it is valid only in ADE case. Moreover, in this case,

$$P_{m,m'}(t) \in t^{-1}\mathbb{Z}_{\geq 0}[t^{-1}] \text{ (positivity)}.$$
(S1) $\overline{L_t(m)} = L_t(m)$ (S2) $M_t(m) = L_t(m) + \sum_{m' < m} P_{m,m'}(t)L_t(m'), \ P_{m,m'}(t) \in t^{-1}\mathbb{Z}[t^{-1}]$

Remark

The characterization properties (S1) and (S2) provide an inductive algorithm for computing $P_{m,m'}(t)$’s, called Kazhdan-Lusztig algorithm.

Conjecture (Hernandez)

For arbitrary cases, we also have
(1) $\forall m \in B, \ ev_{t=1}(L_t(m)) = \chi_q(L(m)).$ (2) $P_{m,m'}(t) \in t^{-1}\mathbb{Z}_{\geq 0}[t^{-1}]$. 

If Conjecture (1) holds (in particular, in ADE cases), we have

$$[M(m)] = [L(m)] + \sum_{m' < m} P_{m,m'}(1)[L(m')] \text{ in } K(C_\bullet).$$
Let $\mathcal{U}_v^-$ be the negative half of the QEA of type $A_N$ over $\mathbb{Q}(v^{1/2})$.

\[
\mathcal{U}_v^- := \text{the } \mathbb{Q}(v^{1/2})\text{-algebra with generators } \{f_i\}_{i=1,\ldots,N}, \text{ relations } \begin{cases} 
  f_i^2 f_j - (v + v^{-1}) f_i f_j f_i + f_j f_i^2 = 0 & \text{if } |i-j| = 1 \\
  f_i f_j - f_j f_i = 0 & \text{if } |i-j| > 1.
\end{cases}
\]

$\sim A_v[N_{-A_N}] \subset \mathbb{Z}[v^{\pm 1/2}]$-subalg $\mathcal{U}_v^-$ the quantized coordinate algebra.

**Property**

$\mathbb{Q}(v^{\pm 1/2}) \otimes_{\mathbb{Z}[v^{\pm 1/2}]} A_v[N_{-A_N}] \simeq \mathcal{U}_v^- \quad \mathbb{C} \otimes_{\mathbb{Z}[v^{\pm 1/2}]} A_v[N_{-A_N}] \simeq \mathbb{C}[N_{-A_N}]$.

Here $N_{-A_N} := \{(N+1) \times (N+1) \text{ unipotent lower triangular matrices}\}$.

- $\exists v_{v=1} : A_v[N_{-A_N}] \rightarrow \mathbb{C}[N_{-A_N}]$ a $\mathbb{Z}$-algebra homomorphism, called the specialization at $v = 1$.

- $\exists$ an $\mathbb{Z}$-algebra anti-involution $\sigma'$ on $A_v[N_{-A_N}]$, called the (twisted) dual bar involution (e.g. $v^{1/2} \mapsto v^{-1/2}$).

($:= \text{the restriction of the } \mathbb{Z}\text{-algebra anti-involution on } \mathcal{U}_v^- \text{ given by } v^{1/2} \mapsto v^{-1/2}, f_i \mapsto -f_i.$)
Let \( i = (i_1, i_2, \ldots, i_\ell) \) be a reduced word of the longest element \( w_0 \) of the Weyl group \( W_{AN} \cong S_{N+1} \).

(e.g. if \( N = 2 \), then \( i = (1, 2, 1) \) or \( (2, 1, 2) \).)
Dual canonical bases

Let $i = (i_1, i_2, \ldots, i_\ell)$ be a reduced word of the longest element $w_0$ of the Weyl group $W_{AN} \simeq \mathfrak{S}_{N+1}$. Let $\Delta_+$ be the set of positive roots of type $A_N$.

$\leadsto \exists \{ \tilde{F}^{up}(c, i) \mid c \in \mathbb{Z}_{\geq 0}^{\Delta_+} \}$ a $\mathbb{Z}[v^{\pm 1/2}]$-basis of $A_v[N_{AN}]$ depending on $i$, which is an analogue of the (dual) PBW-basis associated to $i$ [Lusztig].

**Theorem (Lusztig, Saito, Kimura)**

- $\exists \tilde{B}^{up} := \{ \tilde{G}^{up}(c, i) \mid c \in \mathbb{Z}_{\geq 0}^{\Delta_+} \}$ a $\mathbb{Z}[v^{\pm 1/2}]$-basis of $A_v[N_{AN}]$ such that
  - (B1) $\sigma'(\tilde{G}^{up}(c, i)) = \tilde{G}^{up}(c, i)$, and
  - (B2) $\tilde{F}^{up}(c, i) = \tilde{G}^{up}(c, i) + \sum_{c'} p_{c, c'}(v) \tilde{G}^{up}(c', i)$ with $p_{c, c'}(v) \in v\mathbb{Z}[v]$.

- $\tilde{B}^{up}$ does not depend on the choice of $i$.

The basis $\tilde{B}^{up}$ is called the (normalized) dual canonical basis.
Theorem (Lusztig (i “adapted”), Kato, McNamara (arbitrary), (O. arbitrary))

\[ p_{c,c'}(v) \in \mathbb{Z}_{\geq 0}[v]. \]

Theorem (Lusztig)

For \( c_1, c_2 \in \mathbb{Z}^\Delta_{\geq 0}^+ \), write

\[ \tilde{G}^{\text{up}}(c_1, i) \tilde{G}^{\text{up}}(c_2, i) = \sum c_{c_1,c_2} \tilde{G}^{\text{up}}(c, i). \]

Then \( c_{c_1,c_2} \in \mathbb{Z}_{\geq 0}[v^{\pm 1/2}]. \)
Assume that $\mathcal{U}_q(\mathcal{L}g)$ is of type $A^{(1)}_N (I = \{1, \ldots, N\})$. Define $J_{Q',A^{(1)}_N}$ by

$$J_{Q',A^{(1)}_N} := \{(i, -i+1-2k) \in I \times \mathbb{Z} \mid k = 0, 1, \ldots, 2n-i-1 \text{ and } i \in I\}.$$
Assume that $\mathcal{U}_q(\mathcal{L}g)$ is of type $A^{(1)}_N (I = \{1, \ldots, N\})$.

Define $J_{Q',A^{(1)}_N}$ by

$$J_{Q',A^{(1)}_N} := \{(i, -i+1-2k) \in I \times \mathbb{Z} \mid k = 0, 1, \ldots, 2n-i-1 \text{ and } i \in I\}.$$

Set

$$B_{Q',A^{(1)}_N} := \left\{ \prod_{(i,r)} Y_{u_{i,r}} \in B \mid u_{i,r} \neq 0 \text{ only if } (i, r) \in J_{Q',A^{(1)}_N} \right\},$$

$$C_{Q',A^{(1)}_N} := \text{the full subcategory of } C_* \text{ such that}$$

$$\text{object : } V \text{ with } [V] \in \sum_{m \in B_{Q',A^{(1)}_N}} \mathbb{Z}[L(m)].$$

**Lemma (Hernandez-Leclerc)**

$C_{Q',A^{(1)}_N}$ is an abelian $\otimes$-subcategory.
Set

\[ K_t(C_{Q'}, A^{(1)}_N) := \sum_{m \in \mathcal{B}_{Q', A^{(1)}_N}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{Q', A^{(1)}_N}} \mathbb{Z}[t^{\pm 1/2}] L_t(m). \]

**Lemma**

\[ K_t(C_{Q'}, A^{(1)}_N) \text{ is a } \mathbb{Z}[t^{\pm 1/2}]\text{-subalgebra of } K_t(C_\cdot). \]

\[ \leadsto K_t(C_{Q'}, A^{(1)}_N) \text{ is called the quantum Grothendieck ring of } C_{Q', A^{(1)}_N}. \]
Hernandez-Leclerc isomorphisms in type $A_N^{(1)}$ (2)

Set

$$K_t(C_{Q',A_N^{(1)}}) := \sum_{m \in \mathcal{B}_{Q',A_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{Q',A_N^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).$$

Write

$$J_{Q',A_N^{(1)}} = \{(i_s, r_s) \mid s = 1, \ldots, \ell(= N(N+1)/2)\} \text{ with } r_1 \geq \cdots \geq r_\ell.$$

$$\rightsquigarrow i_{Q'} := (i_1, i_2, \ldots, i_\ell) \text{ is a reduced word of } w_0 \in W^{A_N}.$$

Remark

The reduced word $i_{Q'}$ depends on the choice of the total ordering on $J_{Q',A_N^{(1)}}$. However, its “commutation class” is uniquely determined.

The following results does not depend on this choice.

This $i_{Q'}$ is “adapted to $Q''$.”
Hernandez-Leclerc isomorphisms in type $A_N^{(1)}$ (2)

Set

$$K_t(C_{Q',A_N^{(1)}}) := \sum_{m \in B_{Q',A_N^{(1)}}} \mathbb{Z}[t^{\pm1/2}] M_t(m) = \sum_{m \in B_{Q',A_N^{(1)}}} \mathbb{Z}[t^{\pm1/2}] L_t(m).$$

$$J_{Q',A_N^{(1)}} = \{(s, r_s) \mid s = 1, \ldots, \ell (= N(N+1)/2)\} \text{ with } r_1 \geq \cdots \geq r_\ell.$$

$\mapsto i_{Q'} := (\nu_1, \nu_2, \ldots, \nu_\ell)$ is a reduced word of $w_0 \in \mathcal{W}_{A_N}$.

In the following example, $i_{Q'} = (1, 2, 1, 3, 2, 4, 1, 3, 2, 1)$ etc.

\[ N = 4 \]

\[ \begin{array}{cccccccc}
4 & & & & & & & \\
& 3 & & & & & & \\
& & 2 & & & & & \\
& & & 1 & & & & \\
0 & -1 & -2 & -3 & -4 & -5 & -6 & \\
\end{array} \]
Hernandez-Leclerc isomorphisms in type $\mathbb{A}^{(1)}_N$ (2)

\[
K_t(C_{Q',A^{(1)}_N}) := \sum_{m \in \mathcal{B}_{Q',A^{(1)}_N}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{Q',A^{(1)}_N}} \mathbb{Z}[t^{\pm 1/2}] L_t(m).
\]

\[
J_{Q',A^{(1)}_N} = \{(\nu_s, r_s) \mid s = 1, \ldots, \ell(= N(N + 1)/2)\} \text{ with } r_1 \geq \cdots \geq r_\ell.
\]

$\mapsto i_{Q'} := (\nu_1, \nu_2, \ldots, \nu_\ell)$ is a reduced word of $w_0 \in W^{A_N}$.

**Theorem (Hernandez-Leclerc)**

There exists a $\mathbb{Z}$-algebra isomorphism

\[
\Phi_A : A_v[N^{-A_N}] \xrightarrow{\sim} K_t(C_{Q',A^{(1)}_N})
\]

given by

\[
\nu^{\pm 1/2} \mapsto t^{\mp 1/2} \quad \widetilde{F}^{up}(c, i_{Q'}) \mapsto M_t(m(c)) \quad \forall c \in \mathbb{Z}^{\Delta^+_+},
\]

here $m(c) = \prod_{k=1}^{\ell} Y^{c(s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k})}_{\nu_k, r_k}$. Moreover,

\[
\Phi_A(\widetilde{G}^{up}(c, i_{Q'})) = L_t(m(c)) \quad \forall c \in \mathbb{Z}^{\Delta^+_+}.
\]
Assume that $\mathcal{U}_q(\mathcal{L}g)$ is of type $B_n^{(1)}$ ($I = \{1, \ldots, n\}$).

Let $\tilde{I} := \{1, \ldots, 2n - 1\}$. Define $\tilde{J}_{Q,B_n^{(1)}}$ by

$$\tilde{J}_{Q,B_n^{(1)}} := \{(i, -i + 2 - 2k) \mid k = 0, \ldots, 2n - 1 - i \text{ and } i = n + 1, \ldots, 2n - 1\}$$

$$\cup \{(n, -n + \frac{3}{2} - k) \mid k = 0, \ldots, 2n - 2\}$$

$$\cup \{(i, -i + 1 - 2k) \mid k = 0, \ldots, 2n - 2 - i \text{ and } i = 1, \ldots, n - 1\}.$$

$$n = 3$$

<table>
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<th>$\tilde{I}$</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{J}_{Q,B_n^{(1)}}$</td>
<td>★</td>
<td>★</td>
<td>★</td>
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<td>★</td>
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</table>

$$\mathbb{Z}$$
Assume that $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ is of type $B_{n}^{(1)}$ ($I = \{1, \ldots, n\}$).

Let $\tilde{I} := \{1, \ldots, 2n - 1\}$. Define $\tilde{J}_{Q,B_{n}^{(1)}}$ by

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$$\cup \{(n, -n + \frac{3}{2} - k) \mid k = 0, \ldots, 2n - 2\}$$

$$\cup \{(i, -i + 1 - 2k) \mid k = 0, \ldots, 2n - 2 - i \text{ and } i = 1, \ldots, n - 1\}.$$
Our isomorphisms (1)

Assume that $\mathcal{U}_q(\mathcal{L}g)$ is of type $B_n^{(1)}$ ($I = \{1, \ldots, n\}$).
Let $\tilde{I} := \{1, \ldots, 2n - 1\}$. Define $\tilde{J}_{Q,B_n^{(1)}}$.

Consider the map $\tilde{I} \to I, \iota \mapsto \bar{\iota} := \begin{cases} 
\iota & \text{if } \iota \leq n, \\
2n - \iota & \text{if } \iota > n.
\end{cases}$ “folding”

Set

$$B_{Q,B_n^{(1)}} := \left\{ \prod_{(i, r)} Y_{i, r}^{u_{i, r}} \in B \mid \begin{array}{l}
\text{for some } (\iota, s) \in \tilde{J}_{Q,B_n^{(1)}} \\
\text{with } u_{i, r} \neq 0 \text{ only if } (i, r) = (\bar{\iota}, 2s) \end{array} \right\},$$

$$C_{Q,B_n^{(1)}} := \text{the full subcategory of } C_\bullet \text{ such that }$$

$$\text{object : } V \text{ with } [V] \in \sum_{m \in B_{Q,B_n^{(1)}}} \mathbb{Z}[L(m)].$$

Lemma (Oh-Suh, Hernandez-O.)

$C_{Q,B_n^{(1)}}$ is an abelian $\otimes$-subcategory.
Assume that $\mathcal{U}_q(\mathcal{L}g)$ is of type $B_n^{(1)}$ ($I = \{1, \ldots, n\}$).

Let $\tilde{I} := \{1, \ldots, 2n - 1\}$.

Consider the map $\tilde{I} \rightarrow I, i \mapsto \bar{i} := \begin{cases} i & \text{if } i \leq n, \\ 2n - i & \text{if } i > n. \end{cases}$

“folding”

Set

$$B_{Q,B_n^{(1)}} := \left\{ \prod_{(i,r)} Y_{i,r}^{u_{i,r}} \in \mathcal{B} \mid \begin{array}{c} \text{for some } (i, s) \in \tilde{J}_{Q,B_n^{(1)}} \\ \text{if } (i, r) = (\bar{i}, 2s) \end{array} \right\}.$$
Our isomorphisms (2)

Set

\[ K_t(C_{Q,B_n^{(1)}}) := \sum_{m \in B_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in B_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m). \]

Lemma

\[ K_t(C_{Q,B_n^{(1)}}) \text{ is a } \mathbb{Z}[t^{\pm 1/2}] \text{-subalgebra of } K_t(C_{\bullet}). \]

\( \leadsto K_t(C_{Q,B_n^{(1)}}) \text{ is called the quantum Grothendieck ring of } C_{Q,B_n^{(1)}}. \)
Set
\[ K_t(C_{Q,B_n^{(1)}}) := \sum_{m \in B_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in B_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m). \]

Write
\[ \tilde{J}_{Q,B_n^{(1)}} = \{(i_s, r_s) \mid s = 1, \ldots, \ell(= 2n(2n-1)/2)\} \text{ with } r_1 \geq \cdots \geq r_\ell. \]
\[ \leadsto i_Q^{tw} := (i_1, i_2, \ldots, i_\ell) \text{ is a reduced word of } w_0 \in W^{A_{2n-1}}. \]

Remark
The reduced word \( i_Q^{tw} \) depends on the choice of the total ordering on \( J_{Q,B_n^{(1)}} \). However, its “commutation class” is uniquely determined. The following results does not depend on this choice. This \( i_Q^{tw} \) is always “non-adapted”.

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Our isomorphisms (2)

Set

\[ K_t(C_{Q,B_n^{(1)}}) := \sum_{m \in B_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in B_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m). \]

\[ \tilde{J}_{Q,B_n^{(1)}} = \{ (s, r_s) \mid s = 1, \ldots, \ell (= 2n(2n-1)/2) \} \text{ with } r_1 \geq \cdots \geq r_\ell. \]

\[ \tilde{i}_Q^{\text{tw}} := (r_1, r_2, \ldots, r_\ell) \text{ is a reduced word of } w_0 \in W^{A_{2n-1}}. \]

In the following example, \[ \tilde{i}_Q^{\text{tw}} = (1, 2, 3, 1, 4, 3, 2, 5, 3, 1, 4, 3, 2, 3, 1) \]

etc.
Our isomorphisms (2)

\[ K_t(\mathcal{C}_{Q,B_n^{(1)}}) := \sum_{m \in \mathcal{B}_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] M_t(m) = \sum_{m \in \mathcal{B}_{Q,B_n^{(1)}}} \mathbb{Z}[t^{\pm 1/2}] L_t(m). \]

\[ \tilde{J}_{Q,B_n^{(1)}} = \{(i_s, r_s) \mid s = 1, \ldots, \ell(= 2n(2n-1)/2)\} \text{ with } r_1 \geq \cdots \geq r_\ell. \]

\[ \mapsto i_{Q}^{tw} := (i_1, i_2, \ldots, i_\ell) \text{ is a reduced word of } w_0 \in W^{A_{2n-1}}. \]

Theorem (Hernandez-O.)

There exists a \( \mathbb{Z} \)-algebra isomorphism

\[ \Phi_B : A_v[N_{-}^{A_{2n-1}}] \sim K_t(\mathcal{C}_{Q,B_n^{(1)}}) \]

given by

\[ v^{\pm 1/2} \mapsto t^{\mp 1/2} \quad \text{and} \quad \tilde{F}_{\text{up}}(c, i_{Q}^{tw}) \mapsto M_t(m'(c)) \quad \forall c \in \mathbb{Z}_{\Delta+}^{\geq 0}, \]

where \( m'(c) = \prod_{k=1}^{\ell} Y_{i_k,r_k}^{c(s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k})}. \) Moreover,

\[ \Phi_B(\tilde{G}_{\text{up}}(c, i_{Q}^{tw})) = L_t(m'(c)) \quad \forall c \in \mathbb{Z}_{\Delta+}^{\geq 0}. \]
Positivities in $\mathcal{C}_{Q,B_n^{(1)}}$

By our theorem, the positivities of the dual canonical bases $\tilde{B}^{up}$ can be transported to those of $(q,t)$-characters.

**Corollary (Positivity of Kazhdan-Lusztig type polynomials)**

For $m \in \mathcal{B}_{Q,B_n^{(1)}}$, write

$$M_t(m) = \sum_{m' \in \mathcal{B}_{Q,B_n^{(1)}}} P_{m,m'}(t)L_t(m').$$

as before. Then $P_{m,m'}(t) \in \mathbb{Z}_{\geq 0}[t^{-1}]$.

This is the affirmative answer to Conjecture (2) for $\mathcal{C}_{Q,B_n^{(1)}}$.

**Corollary (Positivity of structure constants)**

For $m_1, m_2 \in \mathcal{B}_{Q,B_n^{(1)}}$, write

$$L_t(m_1)L_t(m_2) = \sum_{m \in \mathcal{B}_{Q,B_n^{(1)}}} c^m_{m_1,m_2} L_t(m).$$

Then we have $c^m_{m_1,m_2} \in \mathbb{Z}_{\geq 0}[t^{\pm 1/2}]$. 
Comparison with Kashiwara-Oh

The following remarkable theorem is recently proved by means of the celebrated *generalized quantum affine Schur-Weyl dualities*, which is developed by Kang, Kashiwara, Kim and Oh:

**Theorem (Kashiwara-Oh ’17)**

There exists a $\mathbb{Z}$-algebra isomorphism

$$[\mathcal{F}]: \text{ev}_{v=1}(A_v[N_{-2}^{A_2}]) \xrightarrow{\sim} K(C_{Q,B_{n}^{(1)}})$$

which maps the dual canonical basis $\text{ev}_{v=1}(\text{B}_\text{up})$ specialized at $v = 1$ to the set of classes of simple modules $\{[L(m)] \mid m \in B_{Q,B_{n}^{(1)}}\}$.

**Theorem (Hernandez-O.)**

$$\Phi_B \mid_{v=t=1} = [\mathcal{F}].$$
Comparison with Kashiwara-Oh

Theorem (Hernandez-O.)

\[ \Phi_B \mid_{v=t=1} = [\mathcal{F}] . \]

Remark

Our construction of \( \Phi_B \) does not imply Kashiwara-Oh’s theorem because, a priori,

- \( \Phi_B \mid_{v=t=1} \) maps \( \text{ev}_{v=1}(\tilde{B}^{\text{up}}) \) to \( \{ \text{ev}_{v=1}(L_t(m)) \mid m \in B_{Q,B_n^{(1)}} \} \), but

- \( [\mathcal{F}] \) maps \( \text{ev}_{v=1}(\tilde{B}^{\text{up}}) \) to \( \{ [L(m)] \mid m \in B_{Q,B_n^{(1)}} \} \),

(The coincidence of these images is nothing but Hernandez’s conjecture (1) !) Hence our result and Kashiwara-Oh’s result are independent.

Our comparison theorem above is proved by looking at the images of dual PBW-bases.
Comparison with Kashiwara-Oh

**Theorem (Kashiwara-Oh ’17)**

There exists a $\mathbb{Z}$-algebra isomorphism

\[
[\mathcal{F}] : \text{ev}_{v=1}(A_v[N_{A2n-1}]) \xrightarrow{\sim} K(C_{Q,B_n^{(1)}})
\]

which maps the dual canonical basis $\text{ev}_{v=1}(\tilde{B}_{up})$ specialized at $v = 1$ to the set of classes of simple modules $\{[L(m)] \mid m \in B_{Q,B_n^{(1)}}\}$.

**Theorem (Hernandez-O.)**

\[
\Phi_B \mid_{v=t=1} = [\mathcal{F}].
\]

**Corollary**

\[
\chi_q(L(m)) = \text{ev}_{t=1}(L_t(m)), \forall m \in B_{Q,B_n^{(1)}}.
\]

This is the affirmative answer to Conjecture (1) for $C_{Q,B_n^{(1)}}$. 
There are several variants in the choices of the subcategories $C_{Q',A_N^{(1)}}$ and $C_{Q,B_n^{(1)}}$. However the parallel results hold. (The choice in this talk is the case that $Q'$ and $Q$ are “equioriented”.)

By combining our $\Phi_B$ with $\Phi_A$ for $A_{2n-1}^{(1)}$, we can obtain a $\mathbb{Z}[v^{\pm1/2}]$-algebra isomorphism $K_t(C_{Q',A_{2n-1}^{(1)}}) \simeq K_t(C_{Q,B_n^{(1)}})$. This isomorphism preserves the set of $(q,t)$-characters of simple modules. (It does not preserve the set of $(q,t)$-characters of standard modules.)

For the choices of $C_{Q',A_{2n-1}^{(1)}}$ and $C_{Q,B_n^{(1)}}$ in this talk, we know explicit correspondence of simple modules in terms of highest monomials.
$T$-system

For $i \in I$, $r \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$, set $m_{k,r}^{(i)} := \prod_{s=1}^{k} Y_{i,r+2i(s-1)}$. ($m_{1,r}^{(i)} = Y_{i,r}$)

The quantum $T$-system of type B [Hernandez-O.]

$\exists \alpha, \beta \in \mathbb{Z}$ such that the following identity holds in $K_t(C_{Q,B_n^{(1)}})$:

$$L_t(m_{k,r}^{(i)}) L_t(m_{k,r+2i}^{(i)}) = t^{\alpha/2} L_t(m_{k+1,r}^{(i)}) L_t(m_{k-1,r+2i}^{(i)}) + t^{\beta/2} S_{k,r,t}^{(i)}.$$  

Here, $S_{k,r,t}^{(i)} = \begin{cases} 
L_t(m_{k,r+2}^{(i-1)}) L_t(m_{k,r+2}^{(i+1)}) & \text{if } i \leq n-2, \\
L_t(m_{k,r+2}^{(n-2)}) L_t(m_{2k,r+1}^{(n)}) & \text{if } i = n-1, \\
L_t(m_{s,r+1}^{(n-1)}) L_t(m_{s,r+3}^{(n-1)}) & \text{if } i = n \text{ and } k = 2s \text{ is even}, \\
L_t(m_{s+1,r+1}^{(n-1)}) L_t(m_{s,r+3}^{(n-1)}) & \text{if } i = n \text{ and } k = 2s + 1 \text{ is odd}. 
\end{cases}$  

$L_t(m_{*,*}^{(0)}) := 1$.

Example ($B_3^{(1)}$-case)

- $L_t(m_{2,r}^{(1)}) L_t(m_{2,r+4}^{(1)}) = t L_t(m_{3,r}^{(1)}) L_t(m_{1,r+4}^{(1)}) + L_t(m_{2,r+2}^{(2)})$.
- $L_t(m_{3,r}^{(3)}) L_t(m_{3,r+2}^{(3)}) = t^{1/2} L_t(m_{4,r}^{(3)}) L_t(m_{2,r+2}^{(3)}) + t^{-1/2} L_t(m_{2,r+1}^{(2)}) L_t(m_{1,r+3}^{(2)})$. 

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Comments on further results and proofs (2)

Sketch of the proof of the existence of $\Phi_B$

0) We have

- $K_t(C_{Q,B_n^{(1)}})$ “truncates” the quantum torus of finitely many variables.
- $A_v[N_{A_{2n-1}}^{-}] \hookrightarrow$ the quantum torus arising from the “quantum initial seed” associated with $i_{Q}^{tw}$ (← quantum cluster algebra).

1) Prove the isomorphism between ambient tori in Step 0. (Here we also use the cluster algebraic observation “$A_{i,r}$’s are $\hat{Y}$-variables”)

2) Show the coincidence between quantum $T$-system and quantum determinantal identities (← mutation sequence. Every algebra generator appears as a cluster variable in this sequence).

Reference: arXiv:1803.06754v1