On some reducible representations of the quantized coordinate algebras

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Main objects: Representations of quantized coordinate algebras for finite dimensional simple Lie algebras over $\mathbb{C}$.

For any sequence $\mathbf{i}$ of the indices of simple roots, we can construct a module $V_{\mathbf{i}}$ over the quantized coordinate algebra.

**Theorem (Soibelman (’90))**

The module $V_{\mathbf{i}}$ is irreducible if and only if $\mathbf{i}$ is a reduced word of an element of the Weyl group.

(Naive) problem: Describe the structure of $V_{\mathbf{i}}$ in the case that $\mathbf{i}$ does not correspond to a reduced word.
How do we “describe” $V_i$?

Key: Kuniba-Okado-Yamada’s result (’13);
There exists a “natural” (but somewhat mysterious) isomorphism between

- the module $V_{i_0}$ corresponding to the longest element $w_0$, and
- (the positive half of) the quantized enveloping algebra $U^+$.

Through this result, we can say that

“the structure of $V_{i_0}$ is described by the algebra structure of $U^+$”.

⇝ How about the other $i$’s?

Our target: The modules $V_{\tilde{i}}$ for $\tilde{i} = (i_l, \ldots, i_1, i_1, \ldots, i_l)$ with $(i_1, \ldots, i_l)$ is a reduced word of an element of the Weyl group. (In particular, they include the case $V_{(i,i)}$.)
Quantized enveloping algebras

Basic notation: Let

- \( \mathfrak{g} \) a symmetrizable Kac-Moody Lie algebra (⊂ finite dimensional simple Lie algebra) over \( \mathbb{C} \),
- \( P \) the weight lattice of \( \mathfrak{g} \) and \( \{ \alpha_i^{(\vee)} \}_{i \in I} \) the simple (co)roots of \( \mathfrak{g} \),
- \( U_q (:= U_q (\mathfrak{g})) \) the quantized enveloping algebra over \( \mathbb{Q}(q) \);
- Chevalley generator: \( E_i, F_i \ (i \in I), \ K_h \ (h \in P^*) \),
- Some relations: \( K_h E_i = q^{\langle \alpha_i, h \rangle} E_i K_h, q\text{-Serre relations,} \ldots \)
  (This is a \( q \)-analogue of \( U(\mathfrak{g}) \)),
- \( U_q^+ \) (resp. \(-\)) the subalgebra of \( U_q \) generated by \( E_i \)'s (resp. \( F_i \)'s).

Hopf algebra structure of \( U_q \) (\( K_i := K_{\frac{\langle \alpha_i, \alpha_i \rangle}{2}}^{\alpha_i^{(\vee)}} \))

\[
\Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i, \quad \Delta(K_h) = K_h \otimes K_h,
\]
\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_h) = 1, \quad \exists \text{antipode}.
\]
Quantized coordinate algebras

Definition

The quantized coordinate algebra \( A_q[\mathfrak{g}] \) is a subalgebra of \( U_q^* \) generated (in fact, spanned) by the matrix coefficients

\[
c^\lambda_{f,v} := c_{f,v}^{V(\lambda)} := (X \mapsto \langle f, X.v \rangle),
\]

here

- \( \lambda \in P_+ := \{ \mu \in P \mid \langle \mu, \alpha_i^\vee \rangle \geq 0 \text{ for all } i \in I \} \),
- \( V(\lambda) \) the integrable highest weight \( U_q \)-module with highest weight \( \lambda \),
- \( v \in V(\lambda), f \in V(\lambda)^{gr} \) (the graded dual of \( V(\lambda) \)).

In \( A_q[\mathfrak{g}] \), we have

\[
c^\lambda_{f,v} c^\lambda_{f',v'} = c^{V(\lambda) \otimes V(\lambda')}_{f \otimes f', v \otimes v'}.\]
An important example

The quantized coordinate algebra $A_q[\mathfrak{sl}_2]$ of $\mathfrak{sl}_2$ is isomorphic to the $\mathbb{Q}(q)$-algebra generated by $c_{ij}$ ($i, j \in \{1, 2\}$) with the following relations; (If “$q = 1$”, then the first six relations are the same.)

\[
\begin{align*}
    c_{i1}c_{i2} &= qc_{i2}c_{i1} \quad (i = 1, 2), \\
    [c_{12}, c_{21}] &= 0, \\
    c_{1j}c_{2j} &= qc_{2j}c_{1j} \quad (j = 1, 2), \\
    [c_{11}, c_{22}] &= (q - q^{-1})c_{12}c_{21}, \\
    c_{11}c_{22} - qc_{12}c_{21} &= 1.
\end{align*}
\]

Indeed the isomorphism of algebras is given by

\[
c_{ij} \mapsto \left. C_{V(\varpi)}^{f_{i},v_{j}} \right|,
\]

where $V(\varpi)$ is an irreducible 2-dimensional representation of $U_q(\mathfrak{sl}_2)$ with a highest weight vector $v_1$ and $v_2 = F.v_1$, and $\{f_1, f_2\}$ is its dual basis.
We have an $A_q[\mathfrak{sl}_2]$-module $V\star := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) |m\rangle$ given by

\[
\begin{align*}
c_{11} : |m\rangle & \mapsto \begin{cases} 
0 & \text{if } m = 0, \\
|m - 1\rangle & \text{if } m \in \mathbb{Z}_{>0},
\end{cases} \\
c_{12} : |m\rangle & \mapsto -q^{m+1} |m\rangle & \text{for } m \in \mathbb{Z}_{\geq 0}, \\
c_{21} : |m\rangle & \mapsto q^m |m\rangle & \text{for } m \in \mathbb{Z}_{\geq 0}, \\
c_{22} : |m\rangle & \mapsto (1 - q^{2(m+1)}) |m + 1\rangle & \text{for } m \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

By its construction, it is easy to see that this is an irreducible $A_q[\mathfrak{sl}_2]$-module.
The module $V_i$ of $A_q[\mathfrak{g}]$

For $i \in I$, we denote by $U_{q_i,i}$ the Hopf subalgebra of $U_q$ generated by $\{E_i, F_i, K_i^{\pm 1}\}$.

The Hopf algebra $U_{q_i,i}$ is isomorphic to $U_{q_i}(\mathfrak{sl}_2)$ ($q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}}$).

We can regard $\mathbb{Q}(q) \otimes \mathbb{Q}(q_i) A_{q_i}[\mathfrak{sl}_2]$ as a subalgebra of $U_{q_i,i}^*$ and denote this subalgebra by $A_i$.

The irreducible module $V_\star$ corresponding to $A_i$ will be denoted by $V_i := \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) |m\rangle_i$.

For $i = (i_1, \ldots, i_l) \in I^l$, set $m_i : U_{q_{i_1},i_1} \otimes \cdots \otimes U_{q_{i_l},i_l} \xrightarrow{\text{multiplication}} U_q$.

Existence of the algebra homomorphism $m_i^* : A_q[\mathfrak{g}] \rightarrow A_{i_1} \otimes \cdots \otimes A_{i_l}$.

Via $m_i^*$, the $\mathbb{Q}(q)$-vector space $V_i := V_{i_1} \otimes \cdots \otimes V_{i_l}$ is regarded as an $A_q[\mathfrak{g}]$-module.
The case: i is a reduced word

Let $W$ be the Weyl group of $\mathfrak{g}$. For $w \in W$, denote by $I(w)$ the set of the reduced words of $w$. (ex. $\mathfrak{g} = \mathfrak{sl}_3$, $w_0$ the longest element, $I(w_0) = \{(1, 2, 1), (2, 1, 2)\}$.)

**Theorem (Soibelman, Narayanan, Tanisaki, (O-))**

The $A_q[\mathfrak{g}]$-module $V_i$ is irreducible if and only if $i \in I(w)$ for some $w \in W$.

Moreover, for $i_1, i_2 \in I(w)$, there is an isomorphism of $A_q[\mathfrak{g}]$-modules $V_{i_1} \rightarrow V_{i_2}$ given by

$$|0\rangle_{i_1} \mapsto |0\rangle_{i_2}.$$

Hence we denote the module $V_i$ ($i \in I(w)$) by $V_w$. (i. e. The modules $\{V_i\}_{i \in I(w)}$ are identified via the above isomorphism.)
KOY’s isomorphism: preliminaries

**Definition (The quantum nilpotent subalgebras \( U_q(w) \))**

For \( w \in W \) and \( i = (i_1, \ldots, i_l) \in I(w) \), set

\[
E_i^c := E_i^{(c_l)} T_{i_l,1}(E_i^{(c_{l-1})}) \cdots T_{i_l,1} \cdots T_{i_2,1}(E_i^{(c_1)}) \text{, and}
\]

\[
F_i^c := F_i^{(c_l)} T_{i_l,1}''(F_i^{(c_{l-1})}) \cdots T_{i_l,1}'' \cdots T_{i_2,1}''(F_i^{(c_1)}) ,
\]

where \( c = (c_1, \ldots, c_l) \in \mathbb{Z}_{\geq 0}^l \), \( X_i^{(c)} \) denotes the divided power and \( T'_{i_l,1}, T''_{i_l,1} \) are \( q \)-analogues of the actions of the braid group.

The set \( \{ E_i^c (\text{resp. } F_i^c) \}_c \) is a linearly independent set of \( U_q^+(\text{resp. } -) \).

Let us denote by \( U_q^+(\text{resp. } -)(w) \) the \( \mathbb{Q}(q) \)-vector subspace of \( U_q^+(\text{resp. } -) \) spanned by \( \{ E_i^c (\text{resp. } F_i^c) \}_c \). Here \( U_q^\pm(e) = \mathbb{Q}(q) \subset U_q^\pm \).

In fact these subspaces do not depend on the choice of \( i \in I(w) \).

**Example**

\[ g = \mathfrak{sl}_3, \quad X_{(1,1,2)}^{(1,1,2)} = X_1^{(2)}(X_2X_1 - qX_1X_2)X_2 \quad (X = E, F). \]
KOY’s isomorphism

Let $U_q^\pm(w)\perp$ be the orthogonal complements of $U_q^\pm(w)$ with respect to the Lusztig type bilinear forms on $U_q^\pm$. Then they are left ideals of $U_q^\pm$.

Denote the quotient maps $U_q^\pm \to U_q^\pm / U_q^\pm(w)\perp$ by $X \mapsto [X]_w$. Note that $\{[E^c_i]_w\text{ (resp. } [F^c_i]_w\}\}_{i,c}$ is a basis of $U_q^\pm / U_q^\pm(w)\perp$ respectively.

Theorem (Kuniba-Okado-Yamada, Saito, Tanisaki, (O-))

Let $i \in I(w)$. Define the $\mathbb{Q}(q)$-linear isomorphism

$$\Phi_{KOV}^{+,w} : V_w \to U_q^+ / U_q^+(w)\perp$$

by $|c\rangle_i \mapsto [E^c_i]_w$, where $|c\rangle_i := |c_1\rangle_{i_1} \otimes \cdots \otimes |c_l\rangle_{i_l}$. Then $\Phi_{KOV}^{+,w}$ is well-defined. (i.e. does not depend on the choice of $i \in I(w)$.)

Moreover, when we identify $V_w$ with $U_q^+ / U_q^+(w)\perp$ via $\Phi_{KOV}^{+,w}$, the multiplication operator $E_i$. is explicitly written in terms of the matrix coefficients.
Towards the non-reduced word case

Technical modification:
\( \tilde{U}_q^0 := \) the group algebra of \( P \) over \( \mathbb{Q}(q) \) (\( =: \bigoplus_{\lambda \in P} \mathbb{Q}(q)K_\lambda \)).

Set \( \tilde{U}_q^{\geq 0} := U_q^\pm \tilde{U}_q^0 \). (i.e. \( K_\lambda E_i = q^{(\alpha_i, \lambda)} E_i K_\lambda \) etc.)

Note that \( U_q^\pm / U_q^\pm (w)^\perp \) can be regarded as \( \tilde{U}_q^{\geq 0} \)-modules by \( K_\lambda [1]_w = [1]_w \) for all \( \lambda \in P \).

Key Tool

There exists an embedding \( \Omega : A_q[\mathfrak{g}] \rightarrow \tilde{U}_q^{\leq 0} \otimes \tilde{U}_q^{\geq 0} \) of \( \mathbb{Q}(q) \)-algebras using the Drinfeld pairing (cf. Joseph’s textbook).
Let $\mathfrak{g} = \mathfrak{sl}_2$. The weight lattice of $\mathfrak{sl}_2$ is written as $P = \mathbb{Z} \varpi$ ($\varpi$ is the fundamental weight). Then

\begin{align*}
\Omega(c_{11}) &= K_{-\varpi} \otimes K_{\varpi}, \\
\Omega(c_{12}) &= (1 - q^2)K_{-\varpi} \otimes EK_{\varpi}, \\
\Omega(c_{21}) &= (1 - q^2)FK_{-\varpi} \otimes K_{\varpi}, \\
\Omega(c_{22}) &= (1 - q^2)^2FK_{-\varpi} \otimes EK_{\varpi} + K_{\varpi} \otimes K_{-\varpi}.
\end{align*}

(Note that $K_{\varpi}E = qEK_{\varpi}$, $K_{-\varpi}F = q^{-1}FK_{\varpi}$)

$S := \{c_{11}^n\}_{n \geq 0}$ is an Ore set in $A_q[\mathfrak{sl}_2]$ and $A_q[\mathfrak{sl}_2][S^{-1}]$ is isomorphic to $\bigoplus_{n \in \mathbb{Z}} U_q^- K_{-n\varpi} \otimes U_q^+ K_{n\varpi}$. 
The case: \(i\) is of the form \(\tilde{i}\)

Via \(\Omega\), the \(\tilde{U}_q^{\leq0} \otimes \tilde{U}_q^{\geq0}\)-module \(U^-/U_q^-(w) \otimes U^+/U_q^+(w)\) can be regarded as an \(A_q[\mathfrak{g}]\)-module.

**Theorem (O-)**

Let \(w \in I(w)\). Define \(\tilde{V}_w := (V_{w-1} \otimes V_w)^{tw}\) (\(\leftarrow\) slightly twisted). Then \(\tilde{V}_w\) is isomorphic to \(U_q^-/U_q^-(w) \otimes U^+/U_q^+(w)\) as an \(A_q[\mathfrak{g}]\)-module. This isomorphism is given by \(\langle 0 \rangle_{w-1} \otimes \langle 0 \rangle_w \mapsto [1]_{w-1} \otimes [1]_w\).

Set \(S := \{c_{f, \lambda}^{\lambda, v, \lambda}\}_{\lambda \in P_+}\) (\(f, \lambda, v\) are highest weight vectors of \(V(\lambda)_{gr}, V(\lambda)\) respectively with \(\langle f, \lambda, v, \lambda \rangle = 1\)). Then the action of \(A_q[\mathfrak{g}]\) on \(\tilde{V}_w\) is extended to \(A_q[\mathfrak{g}][S^{-1}]\).

In particular, the \(A_q[\mathfrak{g}]\)-module \(\tilde{V}_w\) is indecomposable, generated by \(\langle 0 \rangle_{w-1} \otimes \langle 0 \rangle_w\), and decomposed into finite dimensional weight spaces with respect to the action of \(S\). Moreover any nonzero vector generates an infinite dimensional submodule (if \(w \neq e\)).
**Remark**

The isomorphism $\tilde{\Psi}^w_{\text{KOY}} : \tilde{V}_w \rightarrow U_q^-/U_q^-(w) \perp \otimes U_q^+/U_q^+(w) \perp$ in the theorem is NOT equal to $|c_{\text{rev}}\rangle_{i_{\text{rev}}} \otimes |c'\rangle_i \mapsto [F^c_i]_w \otimes [E^c_i]_w$. Indeed

$$\tilde{\Psi}^w_{\text{KOY}}(|c_{\text{rev}}\rangle_{i_{\text{rev}}} \otimes |c'\rangle_i) = [F^c_i]_w \otimes [E^c_i']_w + \sum_{Y \in U^-, X \in U^+, \text{homogeneous}} [Y]_w \otimes [X]_w.$$

We can compute this difference in the case when $g$ is of finite type and $w$ is the longest element $w_0$.

**Example**

Let $g = \mathfrak{sl}_2$. Then

$$\tilde{\Psi}^w_{\text{KOY}}(|2\rangle \otimes |1\rangle) = F^{(2)} \otimes E + \frac{1}{q - q^3} F \otimes 1.$$
From now on we assume that $\mathfrak{g}$ is of finite type.

Let $U'_q$ be a variant of the quantized enveloping algebra whose Cartan part is the group algebra of the root lattice ($\subset P$).

Then we can consider the Drinfeld double $A_q[\mathfrak{g}] \triangleright \triangleleft U'_q$ of $A_q[\mathfrak{g}]$ and $U'_q$. (The $\mathbb{Q}(q)$-algebra $A_q[\mathfrak{g}] \triangleright \triangleleft U_q$ contains $A_q[\mathfrak{g}]$ and $U_q$ as subalgebras.)

Then it is known that $\exists$ an embedding $A_q[\mathfrak{g}] \triangleright \triangleleft U_q$ of algebras, whose restriction to $A_q[\mathfrak{g}]$ coincides with $\Omega$.

If the action of $\tilde{U}_q^{\leq 0} \otimes \tilde{U}_q^{\geq 0}$ on $U_q^-/U_q^-(w)^\perp \otimes U_q^+/U_q^+(w)^\perp$ extends to the $\tilde{U}_q \otimes \tilde{U}_q$-module structure, the $A_q[\mathfrak{g}]$-module structure on $\tilde{V}_w$ extends to the $A_q[\mathfrak{g}] \triangleright \triangleleft U_q'$-module structure!
Relation to the Drinfeld double

**Theorem (O-)**

Let $J$ be a subset of $I$ and $W_J$ the subgroup of $W$ generated by $\{s_j\}_{j \in J}$. Write the longest element of $W$ (resp. $W_J$) as $w_0$ (resp. $w_{J,0}$). Then the $A_q[\mathfrak{g}]$-module structure on $\tilde{V}_{w_0w_{J,0}}$ can be extended to the $A_q[\mathfrak{g}] \bowtie U'_q$-module structure.

In other words, when $w = w_0w_{J,0}$, the $A_q[\mathfrak{g}]$-module $\tilde{V}_{w_0w_{J,0}}$ admits a “compatible $U'_q$-action”.

⇝ Can we “understand” the modules of this type conceptually...? (using quantum flag manifolds...?)