Exercise 14-1

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Exercise Calculate the Fourier transform of the following function $f(x)$.

$$f(x) = \begin{cases} x & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \mathrm{d}x \\
&= \int_0^L x e^{-i\omega x} \mathrm{d}x \\
&= \left[x \frac{e^{-i\omega x}}{-i\omega} \right]_0^L - \int_0^L \frac{e^{-i\omega x}}{-i\omega} \mathrm{d}x \\
&= \frac{L e^{-i\omega L}}{-i\omega} - \left[\frac{e^{-i\omega x}}{-\omega^2} \right]_0^L \\
&= \frac{L e^{-i\omega L}}{-i\omega} + \left[\frac{e^{-i\omega x}}{\omega^2} \right]_0^L \\
&= i \frac{L e^{-i\omega L}}{\omega} + \frac{e^{-i\omega L} - 1}{\omega^2} \\
&= i \frac{\omega L e^{-i\omega L}}{\omega^2} + \frac{e^{-i\omega L} - 1}{\omega^2} \\
&= \frac{i\omega L e^{-i\omega L} + e^{-i\omega L} - 1}{\omega^2} \\
&= \frac{e^{-i\omega L}(1 + i\omega L) - 1}{\omega^2} \\
&= \frac{e^{-iL\omega}(1 + iL\omega) - 1}{\omega^2}
\end{aligned}
$$

Comment The inverse Fourier transform of $F(w)$ obtained above is as follows.

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iL\omega}(1 + iL\omega) - 1}{\omega^2} e^{i\omega x} \mathrm{d}\omega = \begin{cases} x & 0 < x < L \\ L/2 & x = L \\ 0 & \text{otherwise} \end{cases}$$

Note that I do not obtain the inverse Fourier transform by calculating the above integral but obtained it from $f(x)$ by the following equation (Refer to Theorem 1 in the previous document).

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \frac{f(x + 0) + f(x - 0)}{2}$$