An introduction to discrete Fourier transforms

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This document is largely based on the reference book [1] with some parts slightly changed.

1 Discrete Fourier Transform (DFT)

Let $f(x)$ be a periodic function of period 2π. Assume that $f(x)$ is given only in terms of values at the following N points on the range $[0, 2\pi]$:

$$x_l = \frac{2\pi l}{N} \qquad (l = 0, 1, \ldots, N-1). \tag{1}$$

We say that $f(x)$ is being **sampled** at these points. We now would like to find a linear combination of complex exponential functions $\{e^{ikx} | 0 \le k \le N-1\}$

$$\sum_{k=0}^{N-1} F_k e^{ikx}$$

that **interpolates** $f(x)$ at the nodes (1).

$$f(x_l) = \sum_{k=0}^{N-1} F_k e^{ikx_l} \qquad (l = 0, 1, \ldots, N-1)$$

Let $f_l = f(x_l)$. Then we would like to find the coefficients $F_0, \ldots, F_{N-1}$ such that the following equation holds.

$$f_l = \sum_{k=0}^{N-1} F_k e^{ikx_l} \qquad (l = 0, 1, \ldots, N-1) \tag{2}$$

We multiply the both sides of the equation (2) by e^{-imx_l}, where $0 \le m \le N-1$, and sum over l from 0 to $N-1$.

$$\begin{aligned}
\sum_{l=0}^{N-1} f_l e^{-imx_l} &= \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} F_k e^{ikx_l} e^{-imx_l} \\
&= \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} F_k e^{i(k-m)x_l} \\
&= \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} F_k e^{i(k-m)2\pi l/N} \\
&= \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} F_k e^{i(k-m)2\pi l/N} \\
&= \sum_{k=0}^{N-1} F_k \sum_{l=0}^{N-1} e^{i(k-m)2\pi l/N}
\end{aligned}$$

Let $r = e^{i(k-m)2\pi/N}$. Then

$$e^{i(k-m)2\pi l/N} = \left(e^{i(k-m)2\pi/N}\right)^l = r^l.$$

So the above sum is written as follows.

$$\sum_{l=0}^{N-1} f_l e^{-imx_l} = \sum_{k=0}^{N-1} F_k \sum_{l=0}^{N-1} r^l$$

When $k = m$, we have $r = e^0 = 1$, so the sum $\sum_{l=0}^{N-1} r^l$ is calculated as follows.

$$\sum_{l=0}^{N-1} r^l = \sum_{l=0}^{N-1} 1 = N$$

When $k \neq m$, we have $r \neq 1$, so the sum $\sum_{l=0}^{N-1} r^l$ is calculated as follows

$$\sum_{l=0}^{N-1} r^l = \frac{1-r^N}{1-r} = 0,$$

since

$$r^N = \left(e^{i(k-m)2\pi/N}\right)^N = e^{i(k-m)2\pi} = 1.$$

So we obtain the following equality.

$$F_k \sum_{l=0}^{N-1} r^l = \begin{cases} F_m N & k = m \\ 0 & k \neq m \end{cases}$$

So we obtain

$$\sum_{k=0}^{N-1} F_k \sum_{l=0}^{N-1} r^l = F_m N.$$

Since $\displaystyle\sum_{l=0}^{N-1} f_l e^{-imx_l} = \sum_{k=0}^{N-1} F_k \sum_{l=0}^{N-1} r^l$ we obtain

$$\sum_{l=0}^{N-1} f_l e^{-imx_l} = F_m N.$$

By dividing by N we obtain

$$F_m = \frac{1}{N} \sum_{l=0}^{N-1} f_l e^{-imx_l}.$$

By writing k for m we obtain

$$F_k = \frac{1}{N} \sum_{l=0}^{N-1} f_l e^{-ikx_l} = \frac{1}{N} \sum_{l=0}^{N-1} f_l e^{-i2\pi kl/N} \qquad k = 0, \ldots, N-1 \tag{3}$$

since $x_l = \frac{2\pi l}{N}$. The sequence $F_0, \ldots, F_{N-1}$ is called the **discrete Fourier transform** of the given signal $f_0, \ldots, f_{N-1}$.

Let $\omega = e^{2\pi i/N}$. Then $e^{-i2\pi kl/N} = \omega^{-lk}$, so

$$F_k = \frac{1}{N} \sum_{l=0}^{N-1} f_l \omega^{-lk} \qquad k = 0, \ldots, N-1.$$

Then the discrete Fourier transform is written in matrix form as follows.

$$\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{pmatrix} = \frac{1}{N} \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}$$

Note that the element of l-th row and k-th column in the matrix is ω^{-lk}.

By the formula (2), we obtain

$$f_l = \sum_{k=0}^{N-1} F_k e^{ikx_l} = \sum_{k=0}^{N-1} F_k e^{i2\pi kl/N} = \sum_{k=0}^{N-1} F_k \omega^{lk} \qquad (l = 0, 1, \ldots, N-1), \quad (4)$$

which gives the transformation from the sequence $F_0, \ldots, F_{N-1}$ to the sequence $f_0, \ldots, f_{N-1}$. It is called the **inverse discrete Fourier transform**. The inverse discrete Fourier transform is written in matrix form as follows.

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix} = \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \cdots & \omega^{N-1} \\ \omega^0 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{(N-1)} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{pmatrix}$$

The inverse discrete Fourier transform of the discrete Fourier transform of a given signal is the signal itself, since the following equation holds.

$$\begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \cdots & \omega^{N-1} \\ \omega^0 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{(N-1)} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix}^{-1}$$

$$= \frac{1}{N} \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)(N-1)} \end{pmatrix}$$

We do not prove this equation. Refer to textbooks like [1]. Note that A^{-1} represents the inverse matrix of A.

Example: the case for $N = 4$.

Calculate the discrete Fourier transform of the following signal.

$$\boldsymbol{f} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Since $N = 4$, $\omega = e^{2\pi i/4} = e^{\pi i/2} = i$ and thus $\omega^{-lk} = i^{-lk}$. So the discrete Fourier transform of $\boldsymbol{f}$ is calculated as follows.

$$\begin{aligned}
\frac{1}{4}\begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ \omega^0 & \omega^{-3} & \omega^{-6} & \omega^{-9} \end{pmatrix} \boldsymbol{f} &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} f_0 + f_1 + f_2 + f_3 \\ f_0 - if_1 - f_2 + if_3 \\ f_0 - f_1 + f_2 - f_3 \\ f_0 + if_1 - f_2 - if_3 \end{pmatrix}
\end{aligned}$$

2 Fast Fourier Transform (FFT)

The discrete Fourier transform is just a multiplication of a matrix to the given sequence of signal. Naively computing the matrix multiplication requires $O(N^2)$ operations. However, the discrete Fourier transform can be done by the **fast Fourier transform (FFT)**, which needs only $O(N \log_2 N)$ operations. FFT utilizes some specific properties of the matrices.

In computing the discrete Fourier transform and the inverse discrete Fourier transform, it is essential to compute the sequence $b_0, \ldots, b_{N-1}$ from any sequence $a_0, \ldots, a_{N-1}$ as follows.

$$b_k = \sum_{l=0}^{N-1} a_l \omega^{kl} \qquad k = 0, \ldots, N-1 \tag{5}$$

Let's check this. In order to compute $f_0, \ldots, f_{N-1}$ from $F_0, \ldots, F_{N-1}$ following (3), we set $a_k = F_k$ in the equation (5) so that we obtain $f_l = b_l$.

In order to compute $F_0, \ldots, F_{N-1}$ from $f_0, \ldots, f_{N-1}$ we rewrite the formula (3) as follows.

$$\frac{1}{N}\sum_{l=0}^{N-1} f_l \omega^{-kl} = \frac{1}{N}\overline{\sum_{l=0}^{N-1} \overline{f_l} \omega^{kl}}$$

Note that $\overline{z}$ is called the complex conjugate of z, defined as follows.

$$\overline{a + bi} = a - bi$$

We can show the above equation by transforming RHS to LHS as follows.

$$
\begin{aligned}
\text{RHS} &= \overline{\frac{1}{N}\sum_{l=0}^{N-1} \overline{f_l}\omega^{kl}} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} \overline{\overline{f_l}\omega^{kl}} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} f_l\overline{\omega^{kl}} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} f_l\overline{\omega}^{kl} \\
&= \frac{1}{N}\sum_{l=0}^{N-1} f_l(\omega^{-1})^{kl} \qquad (\text{since } \overline{\omega} = \omega^{-1}) \\
&= \frac{1}{N}\sum_{l=0}^{N-1} f_l\omega^{-kl} \\
&= \text{LHS}
\end{aligned}
$$

Then we set $a_l = \overline{f_l}$ in (5) so that we obtain $F_k = \dfrac{1}{N}\overline{b_k}$.

Now we consider the cases where N is a number that satisfies

$$N = 2^n$$

for some natural number n. In these cases we can efficiently compute the discrete Fourier transform and the inverse discrete Fourier transform.

When N is an even number, the following equations hold.

$$\omega^{N/2} = -1, \omega^{N/2+1} = -\omega, \omega^{N/2+2} = -\omega^2, \ldots, \omega^{N-1} = -\omega^{N/2-1}$$

We show these equations. Since $\omega = e^{2\pi i/N}$, we obtain

$$\omega^{N/2} = \left(e^{2\pi i/N}\right)^{N/2} = e^{i\pi} = -1$$

and hence

$$\omega^{N/2+k} = \omega^{N/2}\omega^k = -\omega^k.$$

In the following we write $\omega = e^{2\pi i/N}$ by parameterizing N as follows.

$$\omega_N = e^{2\pi i/N}$$

Then the following equation holds when N is an even number.

$$\omega_N^2 = \omega_{N/2}.$$

We show this as follows.

$$\omega_N^2 = (e^{2\pi i/N})^2 = e^{4\pi i/N} = e^{2\pi i/(N/2)} = \omega_{N/2}$$

By defining

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1} = \sum_{l=0}^{N-1} a_l x^l, \tag{6}$$

the formula (5) can be written as follows.

$$b_k = f(\omega_N^k) \qquad (k = 0, \ldots, N-1)$$

So we obtain $b_0, \ldots, b_{N-1}$ by computing $f(1), \ldots, f(\omega_N^{N-1})$. Let us write this computation as $\mathrm{FFT}_N[f(x)]$.

$$\mathrm{FFT}_N[f(x)] = \{f(1), f(\omega_N), f(\omega_N^2), \ldots, f(\omega_N^{N-1})\}$$

where $f(1), f(\omega_N), f(\omega_N^2), \ldots, f(\omega_N^{N-1})$ represent the values to compute. The formula (6) can be rewritten as follows.

$$\begin{aligned} f(x) &= a_0 + a_2 x^2 + a_4 x^4 + \cdots + a_{N-2} x^{N-2} \\ &\quad + x(a_1 + a_3 x^2 + a_5 x^4 + \cdots + a_{N-1} x^{N-2}) \\ &= p(x^2) + x q(x^2) \end{aligned}$$

Here $p(x)$ and $q(x)$ are defined as follows.

$$\begin{aligned} p(x) &= a_0 + a_2 x + a_4 x^2 + \cdots + a_{N-2} x^{N/2-1} \\ q(x) &= a_1 + a_3 x + a_5 x^2 + \cdots + a_{N-1} x^{N/2-1} \end{aligned}$$

Then $\mathrm{FFT}_N[p(x^2)]$ is as follows.

$$\mathrm{FFT}_N[p(x^2)] = \{p(1), p(\omega_N^2), p(\omega_N^4), \ldots, p(\omega_N^{2N-2})\}$$

Here it is suffice to compute the first half of this sequence since the second half is the same as the first half.

$$\mathrm{FFT}_N[p(x^2)] = \{p(1), p(\omega_N^2), p(\omega_N^4), \ldots, p(\omega_N^{N-2})\}$$

Since $\omega_N^2 = \omega_{N/2}$, we obtain

$$\mathrm{FFT}_N[p(x^2)] = \{p(1), p(\omega_{N/2}), p(\omega_{N/2}^2), \ldots, p(\omega_{N/2}^{N/2-1})\}$$

and hence

$$\mathrm{FFT}_N[p(x^2)] = \mathrm{FFT}_{N/2}[p(x)].$$

In the same way, we obtain

$$\mathrm{FFT}_N[q(x^2)] = \mathrm{FFT}_{N/2}[q(x)].$$

By using the result of $\mathrm{FFT}_{N/2}[p(x)]$ and $\mathrm{FFT}_{N/2}[q(x)]$, $f(\omega_N^k)$ for $k = 0, 1, 2, \ldots, N-1$ can be computed as follows.

$$\begin{cases} f(\omega_N^k) &= p(\omega_{N/2}^k) + \omega_N^k q(\omega_{N/2}^k) \qquad k = 0, 1, \ldots, N/2-1 \\ f(\omega_N^{N/2+k}) &= p(\omega_{N/2}^k) - \omega_N^k q(\omega_{N/2}^k) \qquad k = 0, 1, \ldots, N/2-1 \end{cases} \tag{7}$$

So the computation $\mathrm{FFT}[f(x)]$ can be decomposed into two computations $\mathrm{FFT}_{N/2}[p(x)]$ and $\mathrm{FFT}_{N/2}[q(x)]$ and the computation (7). This gives the fast Fourier transform.

A Some equations for complex numbers

Here we show some equations for complex numbers.

Theorem 1 *For any z_1, $z_2 \in \mathbb{C}$ the following equation holds.*

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$$

Proof Let $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c, $d \in \mathbb{R}$. Then

$$\begin{aligned} \mathrm{LHS} &= \overline{z_1 z_2} \\ &= \overline{(a+bi)(c+di)} \\ &= \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i \end{aligned}$$

$$\begin{aligned} \mathrm{RHS} &= \overline{z_1} \cdot \overline{z_2} \\ &= \overline{(a+bi)} \cdot \overline{(c+di)} \\ &= (a-bi)(c-di) \\ &= (ac-bd)-(ad+bc)i \end{aligned}$$

□

Theorem 2 *For any z_1, $z_2 \in \mathbb{C}$ the following equation holds.*

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

Proof Let $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c, $d \in \mathbb{R}$. Then

$$\begin{aligned} \text{LHS} &= \overline{z_1 + z_2} \\ &= \overline{(a + bi + (c + di)} \\ &= \overline{(a + c) + (b + d)i} \\ &= (a + c) - (b + d)i \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \overline{z_1} + \overline{z_2} \\ &= \overline{(a + bi)} + \overline{(c + d)i} \\ &= (a - bi) + (c - di) \\ &= (a + c) - (b + d)i \end{aligned}$$

□

References

[1] Erwin Kreyszig. *Advanced Engineering Mathematics.* John Wiley & Sons Ltd., tenth edition, 2011.