Exercise 3

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Exercise 3 Fit a parabola (a square function) to the function $\cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so that (the half of) the integral of the squares of the distances between them, where the distance is measured in the vertical direction (the y-direction).

Solution Let the function be $f(x) = ax^2 + bx + c$. The half of the integral of the squares of the distances between $f(x)$ and $\cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is given as follows.

$$\begin{aligned} J &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{f(x) - \cos x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x \end{aligned}$$

J takes the minimum value in the point where the partial derivatives of J with respect to a, b, and c are 0.

$$\frac{\partial J}{\partial a} = 0, \quad \frac{\partial J}{\partial b} = 0, \quad \frac{\partial J}{\partial c} = 0$$

Firstly the partial derivative of J with respect to a is calculated as follows.

$$\begin{aligned} \frac{\partial J}{\partial a} &= \frac{\partial}{\partial a}\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\frac{\partial}{\partial a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial a}\{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\}x^2 \mathrm{d}x \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}x^2 \mathrm{d}x \end{aligned}$$

$$\begin{aligned} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^4 + bx^3 + cx^2 - x^2\cos x\}\mathrm{d}x \\ &= a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4\mathrm{d}x + b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3\mathrm{d}x + c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\mathrm{d}x - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos x\mathrm{d}x \end{aligned}$$

Here we calculate each of the integrals. As for x^4 we obtain

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4\mathrm{d}x &= 2\int_0^{\frac{\pi}{2}} x^4\mathrm{d}x \quad \text{(since } x^4 \text{ is an even function)} \\ &= 2\left[\frac{x^5}{5}\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi^5}{80} \end{aligned}$$

As for x^3 its integral is 0 since it is an odd function.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3\mathrm{d}x = 0$$

As for x^2 we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} x^2\mathrm{d}x = 2\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} = 2\cdot\frac{\pi^3}{24} = \frac{\pi^3}{12}$$

As for $x^2\cos x$ we obtain

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos x\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} x^2\cos x\mathrm{d}x \quad \text{(since } x^2\cos x \text{ is an even function)}$$

In the following we calculate $\int_0^{\frac{\pi}{2}} x^2\cos x\mathrm{d}x$.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x^2\cos x\mathrm{d}x &= \left[x^2\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\sin x\mathrm{d}x \\ &= \frac{\pi^2}{4} - 2\int_0^{\frac{\pi}{2}} x\sin x\mathrm{d}x \end{aligned}$$

Here we calculate $\int_0^{\frac{\pi}{2}} x\sin x\mathrm{d}x$.

$$\begin{aligned} \int_0^{\frac{\pi}{2}} x\sin x\mathrm{d}x &= \left[x\frac{\cos x}{-1}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\cos x}{-1}\mathrm{d}x \\ &= [\sin x]_0^{\frac{\pi}{2}} \\ &= 1 \end{aligned}$$

Now we resume the calculation of $\int_0^{\frac{\pi}{2}} x^2 \cos x \mathrm{d}x$.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x^2 \cos x \mathrm{d}x &= \frac{\pi^2}{4} - 2 \cdot 1 \\ &= \frac{\pi^2}{4} - 2\end{aligned}$$

Thus $\frac{\partial J}{\partial a}$ is obtained as follows.

$$\begin{aligned}\frac{\partial J}{\partial a} &= \frac{\pi^5}{80}a + \frac{\pi^3}{12}c - 2(\frac{\pi^2}{4} - 2) \\ &= \frac{\pi^5}{80}a + \frac{\pi^3}{12}c - \frac{\pi^2}{2} + 4\end{aligned}$$

Secondly the partial derivative of J with respect to b is calculated as follows.

$$\begin{aligned}\frac{\partial J}{\partial b} &= \frac{\partial}{\partial b}\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\frac{\partial}{\partial b}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial b}\{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\{ax^2 + bx + c - \cos x\}x \mathrm{d}x \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}x \mathrm{d}x \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^3 + bx^2 + cx - x\cos x\} \mathrm{d}x \\ &= a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3 \mathrm{d}x + b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \mathrm{d}x + c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \mathrm{d}x - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x \mathrm{d}x \\ &= 2b\int_0^{\frac{\pi}{2}} x^2 \mathrm{d}x \quad (x^3,\ x,\ \text{and } x\cos x \text{ are odd functions and } x^2 \text{ is an even function}) \\ &= 2b \cdot \frac{\pi^3}{24} \\ &= \frac{\pi^3}{12}b\end{aligned}$$

Thirdly the partial derivative of J with respect to c is calculated as follows.

$$\frac{\partial J}{\partial c} = \frac{\partial}{\partial c}\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{ax^2 + bx + c - \cos x\}^2 \mathrm{d}x$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{\partial}{\partial c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{ax^2+bx+c-\cos x\}^2 \mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\partial}{\partial c}\{ax^2+bx+c-\cos x\}^2 \mathrm{d}x \\
&= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\{ax^2+bx+c-\cos x\}\cdot 1 \ \mathrm{d}x \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\{ax^2+bx+c-\cos x\}\mathrm{d}x \\
&= a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x^2\mathrm{d}x + b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\mathrm{d}x + c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1 \ \mathrm{d}x - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\mathrm{d}x \\
&= 2a\int_{0}^{\frac{\pi}{2}}x^2\mathrm{d}x + \pi c - 2\int_{0}^{\frac{\pi}{2}}\cos x\mathrm{d}x \\
&= 2a\cdot\frac{\pi^3}{24} + \pi c - 2\left[\sin x\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi^3}{12}a + \pi c - 2
\end{aligned}
$$

Thus we obtain the system of equations with respect to a, b, and c.

$$
\begin{aligned}
\frac{\pi^5}{80}a + \frac{\pi^3}{12}c - \frac{\pi^2}{2} + 4 = 0 \\
\frac{\pi^3}{12}b = 0 \\
\frac{\pi^3}{12}a + \pi c - 2 = 0
\end{aligned}
$$

By solving this, we obtain the solution.

$$
a = \frac{60\pi^2 - 720}{\pi^5}, \quad b = 0, \quad c = \frac{60 - 3\pi^2}{\pi^3}
$$

Hence the function is obtained as follows.

$$
f(x) = \frac{60\pi^2 - 720}{\pi^5}x^2 + \frac{60 - 3\pi^2}{\pi^3}
$$

The function is depicted with $\cos x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ in Fig. 1. In Fig. 1 the red curve is the square function and the green curve is the function $\cos x$.

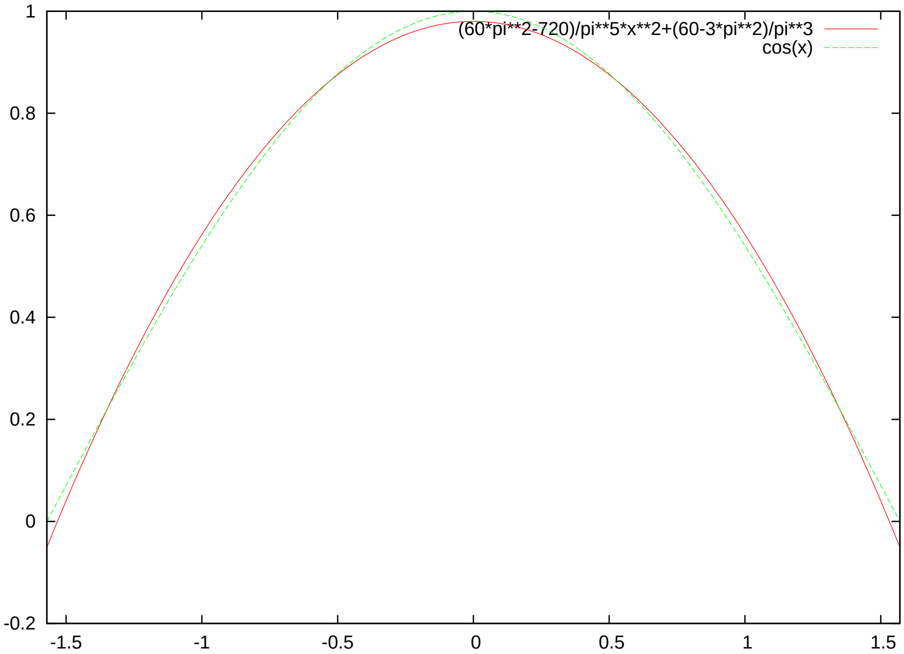

Figure 1: The closest square function to $\cos x$ on the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$